GENERELIZED FUNCTIONS LECTURES

1. Lecture 1-The space of generalized functions on $\mathbb{R}^n$ and operations on them

1.1. **Motivation.** One of the basic examples for a generalized function is the "Dirac Delta function". While it is not a function, δ_t can be described by $\delta_t(x) := \begin{cases} \infty & x = t \\ 0 & x \neq t \end{cases}$, and by satisfying the equality $\int\limits_{-\infty}^{\infty} \delta_t(x)dx = 1$. Notice that $\int\limits_{-\infty}^{\infty} \delta_t(x)f(x)dx = f(t)\int\limits_{-\infty}^{\infty} \delta_t(x)dx = f(t)$. Here are several possible motivations to define generalized functions:

- Every real function $f : \mathbb{R} \to \mathbb{R}$ can be established as a (ill-defined) sum of $\aleph$ indicator functions $f \equiv \sum\limits_{t\in\mathbb{R}} f_t$, where $f_t(x) := \begin{cases} 1 & x = t \\ 0 & x \neq t \end{cases}$.
- Sometimes the solution for a differential equation (or even just the derivative of a function) is not a function, but only a generalized function. Using generalized functions, we can formulate solutions in such cases.
- In physics, Dirac Delta function can describe the density of a point mass.

1.2. **Basic definitions.** We denote by $C_c^\infty(\mathbb{R})$ the space of smooth real functions with compact support.

Definition. A generalized function is a continuous linear functional $\xi : C_c^\infty(\mathbb{R}) \to \mathbb{R}$. We sometimes use the notation $\langle\xi, \phi\rangle$ instead of $\xi(\phi)$.

To define what does "continuous" means we need to define a topology on $C_c^\infty(\mathbb{R})$. This is equivalent to define what is a convergent sequence in $C_c^\infty(\mathbb{R})$ (why? there is something that need to be said here about uniform topology), and then ξ is continuous iff the image of a convergent sequence converges to the image of its limit.

Definition. Given $f \in C_c^\infty(\mathbb{R})$ and a sequence $\{f_n\}_{n\in\mathbb{N}}$ with $f_n \in C_c^\infty(\mathbb{R})$ for all n, we say that $\{f_n\}$ *converges in* $C_c^\infty(\mathbb{R})$ to f if:

1) There exists a compact $K \subset \mathbb{R}$ for which $Supp(f) \cup \bigcup\limits_{n\in\mathbb{N}} Supp(f_n) \subseteq K$.

2) For every order $k = 0,1,2...$, the derivatives $\{f_n^{(k)}\}$ converge uniformly to the derivative $f^{(k)}$.

Recall a function f is $locally - L^1$, denoted $f \in L^1_{LOC}$, if the restriction to any compact subset in its domain is an L^1 function. Given a real function $f \in L^1_{LOC}$ we'll define $\xi_f : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ to be the generalized function $\xi_f(\phi) := \int\limits_{-\infty}^{\infty} f(x) \cdot \phi(x)dx$ (notice the integral converges as it vanishes outside K, and $f|_K, \phi|_K \in L^1$). These are sometimes called *regular generalized functions.*

Exercise. For any $f \in L^1_{LOC}$, ξ_f is a well defined distribution.

The space of generalized real functions is denoted $C^{-\infty}(\mathbb{R}) := C_c^\infty(\mathbb{R}))^*$. Also, we have that $C(\mathbb{R}) \subset L^1_{LOC} \subset C^{-\infty}(\mathbb{R})$, where the second inclusion is derived from the embedding $f \mapsto \xi_f$.

Exercise. Prove that there exists a function $f \in C_c^\infty(\mathbb{R})$ which isn't the zero function. **Hint:** Use function such as $e^{-1/(1-x)^2}$ as your building block.

Definition. We say the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges weakly to f if for every $F \in C_c^\infty(\mathbb{R})$ we have: $\lim\limits_{n\longrightarrow\infty} \int\limits_{-\infty}^{\infty} F(x) \cdot f_n(x)dx = \int\limits_{-\infty}^{\infty} F(x) \cdot f(x)dx$. Now we want to take a completion with respect to this weak convergence, and for this we need the notion of Cauchy sequence: A sequence $\{f_n\}$ is called a *weakly Cauchy sequence* if

$$\forall g \in C_c^\infty(\mathbb{R}),\, \epsilon > 0\, \exists N \text{ such that } \forall m,n > N \int\limits_{-\infty}^{\infty} (f_n(x) - f_m(x))\, g(x)dx < \epsilon.$$

Exercise. There is a natural isomorphism $\overline{C_c^\infty(\mathbb{R})}^w \simeq (C_c^\infty(\mathbb{R}))^*$ as vector spaces.

Definition. A sequence $\phi_n \in C_c(\mathbb{R})$ of continuous, compactly supported functions is said to be an *approximation to the identity* if the ϕ_n are non-negative, have total mass $\int\limits_{-\infty}^{\infty} \phi_n(x) \cdot dx = 1$ and for any fixed r, ϕ_n is supported on $[-r,r]$ for n sufficiently large. One can generate such a sequence by starting with a single non-negative continuous compactly supported function ϕ_1 of total mass 1, and then setting $\phi_n(x) = n\phi_1(nx)$. Many other constructions are possible also.

Notice that given $\eta \in C^{-\infty}(\mathbb{R})$ of the form $\eta = \xi_f$, we can "recover" f completely by applying $\langle \xi_f, \phi_n(x+t)\rangle$, and take the limit to get $f(t)$.

1.3. **Derivatives of generalized functions.** Let $f \in C_c^\infty(\mathbb{R})$. We defined $\xi_f(\phi) := \int\limits_{-\infty}^{\infty} f(x) \cdot \phi(x)dx$, and thus $\xi_{f'}(\phi) = \int\limits_{-\infty}^{\infty} f'(x) \cdot \phi(x)dx$. Using integration by parts we'll get $\xi_{f'}(\phi) = f(x) \cdot \phi(x)|_{-\infty}^{\infty} - \int\limits_{-\infty}^{\infty} f(x) \cdot \phi'(x)dx$. However, since ϕ and f has compact support, we know that $f(x) \cdot \phi(x)|_{-\infty}^{\infty} = 0$. Thus, we'll **define** $\xi'(\phi) := -\xi(\phi')$.

For example, the derivative of δ_0 can be (badly) described as

$$\delta_0'(x) := \begin{cases} \infty & x \to 0^- \\ -\infty & x \to 0^+ \\ 0 & otherwise \end{cases}.$$

This is a bad description, since we can't evaluate generalized functions at specific points (also it's hard to describe δ_0'', δ_0''' this way). When $\delta_0'(x)$ is applied to some $\phi \in C_c^\infty(\mathbb{R})$, according to our definition we'll get $\delta_0'(\phi) = -\delta_0(\phi') = -\phi'(0)$.

Exercise. Find a function $F \in L^1_{LOC}$ for which $F' = \delta$. **Hint:** $F(x) := \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$.

1.4. **The support of generalized functions.** We cannot evaluate a generalized function at a point. Therefore, we cannot just define its support by $Supp(\xi) := \overline{\{x \in \mathbb{R} \mid \xi(x) \neq 0\}}$. However, if for some neighborhood $U \subset \mathbb{R}$ we have $\forall f \in C_c^\infty(U)$, $\xi(f) = 0$, then evidently $supp(\xi) \subseteq U^c$. In this case we'll denote $\xi|_U \equiv 0$.

Notation: $C_c^\infty(U)$ is the space of smooth functions $f : U \to \mathbb{R}$ supported in some compact subset of U. Given a compact subset K of some space X, we denote $C_K^\infty(X)$ the space of smooth functions $f : X \to \mathbb{R}$ with $supp(f) \subseteq K$. In particular $C_K^\infty(X) \subseteq C_c^\infty(X)$ for every $K \subseteq X$.

As another example for a generalized function's support: it's reasonable to expect $Supp(\delta_t) = \{t\}$. So, we'd like to define $Supp(\xi)$ to be the complement of the union over all neighborhoods $U \subset \mathbb{R}$ such that $\forall f \in C_c^\infty(U), \xi(f) = 0$. This definition is well defined only if we solve the following exercise:

Exercise. Let U_1, U_2 be open subsets of $\mathbb{R}$. Show that:

1) if $\xi|_{U_1} \equiv \xi|_{U_2} \equiv 0$ then $\xi|_{U_1 \cup U_2} \equiv 0$. **Hint:** Use partition of unity.

2) Show this also holds for any union of such compact $\{U_i\}_{i \in I}$.

Note that the support of δ_0' is just $\{0\}$ and yet, given some $f \in C(\mathbb{R})$ for which $f(0) = 0, f'(0) \neq 0$, we'll have $\delta_0'(f) = -\delta_0(f') = -f'(0) \neq 0$. In other words, having $f(0) = 0$ isn't enough to get δ_0' to vanish on f. We need f to vanish with all its derivatives.

Exercise. 1) The support of $\delta^{(n)}$ is $\{0\}$ for any n.

2) Find all the generalized functions $\xi \in C_c^{-\infty}(\mathbb{R})$ for which $Supp(\xi) = \{0\}$. **Hint:** All the functions $\delta_0^{(n)}$ for $n \in \mathbb{N}$ and their (finite) linear combinations.

3) $Supp(a\xi_1 + b\xi_2) \subseteq Supp(\xi_1) \cup Supp(\xi_2)$.

4) $Supp(\xi) - Supp(\xi)^\circ \subseteq Supp(\xi') \subseteq Supp(\xi)$.

1.5. Products and convolutions of generalized functions.

Definition. Let $f \in C_c^\infty(\mathbb{R}), \xi \in C_c^{-\infty}(\mathbb{R})$. We'd like to have $(f \cdot \xi)(\phi) = \int\limits_{-\infty}^{\infty} \xi(x) \cdot f(x) \cdot \phi(x) dx$. Thus, we'll define $(f \cdot \xi)(\phi) := \xi(f \cdot \phi)$.

Actually, even though we can multiply every such f and ξ, the product of *two generalized functions* will not always be defined. Notice that indeed in some topologies the product of two Cauchy sequences isn't always a Cauchy sequence.

Recall that given two functions f, g, their *convolution* is the function $(f * g)(x) := \int\limits_{-\infty}^{\infty} f(t) \cdot g(x-t) dt$. The convolution of two smooth functions will always be smooth. In addition, if f, g have compact support, than so will $f * g$.

Exercise. *Supp*$(f * g)$ is the *Minkowski sum* of $Suppf$ and $Suppg$. Therefore $f, g \in C_c^\infty(\mathbb{R})$ implies $f * g \in C_c^\infty(\mathbb{R})$.

Given $f, g \in C_c^\infty(\mathbb{R})$ we can write $(f * g)(x) = \xi_f(\tilde{g_x})$, where $\tilde{g_x}(t) := g(x - t)$. This gives the motivation to define the convolution $\xi * g$ to be the function $(\xi * g)(x) = \xi(\tilde{g_x})$ (notice: the convolution between a function and a generalized function is a function- not a generalized function).

Exercise. Show that for $\phi \in C_c^\infty(\mathbb{R})$ we get that $\xi * \phi$ is a smooth function.

Next is the definition for convolution of two generalized functions. We won't define it for every couple of generalized functions -only for those with compact support, or more precisely, when at least one of the generalized functions have compact support. For $\xi_f, \xi_g \in C_c^{-\infty}(\mathbb{R})$ we'd like to have:

$$(\xi_f * \xi_g)(\phi) = \int\limits_{x=-\infty}^{\infty} (f*g)(x)\cdot\phi(x)dx = \int\limits_{x=-\infty}^{\infty}\int\limits_{t=-\infty}^{\infty} f(t)\cdot g(x-t)\cdot\phi(x)dtdx$$

Rearranging the expression and replacing the order of integration gives:

$$(\xi_f * \xi_g)(\phi) = \int\limits_{t=-\infty}^{\infty} f(t)\int\limits_{x=-\infty}^{\infty} g(x-t)\cdot\phi(x)dxdt$$

In a "usual" convolution, the arguments of the multiplied functions in the integral sum up to the convolution's argument (e.g., $(f*g)(x) := \int\limits_{-\infty}^{\infty} f(t)\cdot g(x-t)dt$, and $x = t + (x-t)$). In our case, we denote $\bar{\phi}(x) := \phi(-x)$, and write:

$$\int\limits_{t=-\infty}^{\infty} f(t)\int\limits_{x=-\infty}^{\infty} g(x-t)\cdot\bar{\phi}(-x)dxdt = \int\limits_{t=-\infty}^{\infty} f(t)\cdot(\xi_g*\bar{\phi})(-t)dt = \xi_f\overline{(\xi_g*\bar{\phi})}$$

Definition. We define$(\xi_f * \xi_g)(\phi) := \xi_f(\overline{(\xi_g*\bar{\phi})})$.

However, some formal justification is required. Given a compact $K \subset \mathbb{R}$, we'll say ρ is a *cutoff function of* K if $\rho|_K \equiv 1, \rho|_V \equiv 0$, when $V \subset \mathbb{R}\backslash K$.

Exercise. Let K, V as above. Show that there always exists a smooth cutoff function. **Hint:** use Urison's Lemma.

Thus, given some $\xi \in C_c^{-\infty}(\mathbb{R})$ with $Supp(\xi) \subset K$ we will have $\xi(\phi) = \xi(\rho_K\cdot\phi)$. This enables us to define ξ as a functional over all $C^\infty(\mathbb{R})$ and not only on $C_c^\infty(\mathbb{R})$). For every $\phi \in C^\infty(\mathbb{R})$ we define $\xi(\phi) = \xi(\rho_K\cdot\phi)$ with $K := supp(\xi) \subset \mathbb{R}$.

Exercise. 1) Show that $(\xi*\eta)' = \xi'*\eta = \xi*\eta'$. **Hint:** First show that $\delta*\eta = \eta$, and that $\delta'*\eta = \eta'$. Then show we have associativity: $\delta'*(\xi*\eta) = (\delta'*\xi)*\eta$.

2) In an exercise above we showed: if $\phi \in C_c^\infty(\mathbb{R})$ then the convolution $\xi*\phi$ is smooth. Show that if ϕ is smooth, and $Supp(\xi)$ is compact, then $\xi*\phi$ will still be smooth.

1.6. **Generalized functions and differential operators.** A differential equation can be described by the equality "$Af = g$", where A is a differential operator. Let's try to solve such an equation, when we assume A is a linear differential operator, and is invariant under translations (i.e., we'll have $A\bar{f} = \overline{Af}$, where $\bar{\phi}$ is any fixed translation of ϕ). An example for such operator is a differential operators with fixed coefficients (e.g., $Af := f'' + 5f' + 6f$).

A simple case is finding G for which the equation $AG = \delta_0$ holds. Given such G, and using A's invariance under translations, we get that $AG_x = \delta_x$, for $G_x(t) := G(t-x)$. We can use the exercise above to show that $A(f*h) = (Af)*h$ for any two functions f, h and then deduce that$A(G * g) = AG * g = \delta_0 * g = g$. Hence, we can find a general solution f for $Af = g$ by solving only one simpler case $AG = \delta_0$. The solution G is called *Green's function of the operator.*

Exercise. 1) Let A be a differential operator with fixed coefficients. Choose any solution for the equation $AG = \delta_0$, and describe the conditions G have to meet without using generalized functions.

2) Without using generalized functions, please explain the equation $A(G * g) = g$ we got for the solution G.

3) Solve the equation $\Delta f = \delta_0$ (where Δ is the Laplacian operator).

1.7. Regularization of generalized functions.

Definition. Let $\{\xi_\lambda\}_{\lambda \in \mathbb{C}}$ be a family of generalized functions. We say the family is *analytic* if $\langle \xi_\lambda, f \rangle$ is analytic (as function of $\lambda \in \mathbb{R}$) for every $f \in C_c^\infty(\mathbb{R})$.

Example. We denote $x_+^\lambda := \begin{cases} x^\lambda & x > 0 \\ 0 & x \leq 0 \end{cases}$, and define the family by $\xi_\lambda := x_+^\lambda$. The behavior of the function changes as λ changes: When $Re(\lambda) > 0$ we'll have a nice continuous function; If $Re(\lambda) = 0$ We'll get a step function and for $Re(\lambda) \in (-1, 0)$, x_+^λ will not be bounded. We'd like to extend the definition analytically for $Re(\lambda) < -1$.

A derivation of x_+^λ (both as a complex function or as defined for a generalized function) gives $\xi'_\lambda = \lambda \cdot \xi_{\lambda-1}$. This is a functional equation, that enables us to **define** $\xi_{\lambda-1} := \frac{\xi'_\lambda}{\lambda}$, and thus extend ξ_λ to every $\lambda \in \mathbb{C}$. This extension isn't analytic, but is meromorphic: it has a pole in $\lambda = 0$, and by the extension formula, in $\lambda = -1, -2, ...$

This is an example for *a meromorphic family of generalized functions.* Let's give a formal definition. Our $\{\xi_\lambda\}_{\lambda \in \mathbb{C}}$ has a set of poles $\{\lambda_n\}$ (poles are always discrete), whose respective orders are denoted $\{d_n\}$. The family will be called *meromorphic* if every pole λ_i has a neighborhood U_i, such that $\langle \xi_\lambda, f \rangle$ is analytic for every $f \in C_c^\infty(\mathbb{R})$ and $\lambda_i \neq \lambda \in U_i$.

Exercise. For the above example $\xi_\lambda := x_+^\lambda$, find the order and the leading coefficient for every pole.

Example. For a given $p \in \mathbb{C}[x_1, ...x_n]$, we denote similarly $p_+(x_1, ...x_n)^\lambda := \begin{cases} p(x_1, ...x_n)^\lambda & x > 0 \\ 0 & x \le 0 \end{cases}$. The problem of finding the meromorphic continuation for a general polynomial was open for a while. It was solved by Bernstein by defining a differential operator $Dp_+^\lambda := b(\lambda) \cdot p_+^{\lambda-1}$, where $b(\lambda)$ was a polynomial pointing on the location of the poles.

Exercise. 1) Solve the problem of finding an analytic continuation for $p_+(x_1, ...x_n)^\lambda$ in the case $p(x, y, z) := x^2 + y^2 + z^2 - a$.

2) Solve the problem of finding an analytic continuation for $p_+(x_1, ...x_n)^\lambda$ in the case $p(x, y, z) := x^2 + y^2 - z^2$.

2. Lecture 2- topological properties of $C_c^\infty(\mathbb{R}^n)$

We want to analyze the space of distributions $C^{-\infty}(\mathbb{R}^n)$ and to define a topology on it. For this we use facts from topological vector spaces.

2.1. **Topological vector spaces.**

Definition. A *topological vector space* (or *linear topological space*) is a linear space with a topology, s.t. multiplication by scalar and vectors addition is continuous. More precisely: there exists continuous operations:

1) $+ : V \times V \longrightarrow V$

2) $\cdot : F \times V \longrightarrow V$, where F is some topological field such that V is a vector space over it.

This demand limits the topology we can have. For example, giving the space discrete topology will force a discrete topology on the field.

Since addition of points is continuous, translation is also continuous. This makes all the points in the space "similar" and therefore the open sets of every point x are the same as those around 0. This property is called homogeneity. We're mainly interested in "nice" topological vector spaces. Specifically: **We assume all the topological vector spaces are Hausdorff**. Note that for a non Hausdorff space V we can quotient by the closure of $\{0\}$ and get a Hausdorff space. This will make sense by the following exercise.

Definition. Let V be a topological vector space over F.

1) We say that a set $A \subseteq V$ is *convex* if for every $a, b \in A$ the linear combination $ta + (1-t)b \in A$ where $t \in [0, 1]$.

2) We say that V is *locally convex* if it has a basis of its topology which consists of convex sets.

3) For every open convex set $0 \in C$ in V we set for any $x \in V$: $N_C(x) = inf\{\alpha \in \mathbb{R}_{\geq 0} : \frac{x}{\alpha} \in C\}$.

4) We say that a set $W \subseteq V$ is *balanced* if $\lambda\lambda W \subseteq W$ for all $|\lambda| \leq 1$ where $\lambda \in F$.

Note that a convex set C is balanced iff it is symmetric ($C = -C$).

Exercise. 1) Find a topological vector space which is not locally convex (not necessarily of finite dimension.

2) Show that any finite dimensional Hausdorff locally convex space is isomorphic to F^n. This is also true for linear topological spaces that are not locally convex, but the proof is harder.

3) Let V be a locally convex linear topological space. Prove that V is Hausdorff iff $\{0\}$ is a closed set.

Remark. From the homogeneity of V, $\{0\}$ is a closed set iff $\forall x \in V\{x\}$ is a closed set. The exercise shows a locally convex linear topological space satisfies the separation axiom T_1 iff it satisfies T_2.

Exercise. Let $0 \in C$ be an open convex set in a topological vector space V .

1) Show that $N_C(x) < \infty$ for all$x \in V$.

2) Show that if furthermore C is balanced then $N_C(x)$ is a semi-norm.

In a locally convex space we have a basis to the topology consisting of convex sets. We can assume all the sets are symmetric: First notice it's enough to show this for open sets around 0 (from homogeneity of the space). Then, given any open convex neighborhood A of 0, we know $A \cap -A$ is a (non-empty) symmetric convex open subset of it. Therefore we have a basis for our topology consisting of symmetric convex sets.

However, there is a bijection between semi-norms on the space and symmetric convex sets. Given a semi-norm N on V, the bijection maps N to its unit ball $\{x \in V \mid N(x) \leq 1\}$ (it's symmetric by absolute homogeneity and convex by the triangle inequality). Note he semi-norm $N_C(x)$ we defined isn't a norm. Specifically, if C contains the subspace $span\{v\}$, we'll get $n_C(v) = 0$ (even though $v \neq 0$). However, given the basis T for our topology, we can not get $n_C(v) = 0$ for **all** the sets $C \in T$. Since in this case we'd have $span\{v\} \subseteq \bigcap\limits_{C \in T} C$, contradicting the Hausdorff assumption.

Definition. A set $C \subseteq V$ is ***absorbent*** $\forall x \in V \, \exists \lambda : \frac{x}{\lambda} \in C$. i.e., multiplying C by a big enough scalar can reach every point in the space. For absorbent $C \subseteq V$ we'll have $N_C(v) < \infty$ for all $v \in V$ directly from definition. Every open set is absorbent, and thus we can define our norm for all the sets in the basis.

Example. The segment $\{(x, 0) \mid x \in [0, 1]\}$ in $\mathbb{R}^2$ isn't absorbent, and for $y = (1, 0)$ we get $n_C(y) = \infty$.

Exercise. 1) Find a locally convex topological vector space V such that V has no continuous norm on it. That is, every convex open set C contains a line $span\{v\}$, so $N_C(v) = 0$.

2) Find a non locally convex space.

In conclusion, a locally convex space possess a basis to the topology consists of collection of sets that defines a system of semi-norms. Some authors use this statement as the definition of locally convex space.

2.2. **Defining completeness.** In a metric space, a point belongs to the closure of a given set if and only if it is the limit of some sequence of points belonging to that set. The convergence of the sequence $(a_n)_{n\in\mathbb{N}}$ to the point x is defined by the requirement that for any $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $d(a_n, x) < \epsilon$ whenever $n \geq N$. This is equivalent to the requirement that for any neighborhood U of x there is some $N \in \mathbb{N}$ such that a_n belongs to U whenever$n \geq N$.

For a general topological vector space V, even though we don't have a metric on V, we can define Cauchy series:

Definition. A series $\{x_n\} \subset V$ is called a *Cauchy series*, if for every neighborhood U of $0 \in V$ there is an index $n_0 \in \mathbb{N}$ such that $m, n > n_0$ implies $x_n - x_m \in U$.

Remark. More generally, if X has a *uniform topology*, then we can define a notion of a Cauchy sequence. We will not give the definition of a uniform topology, but we mark that any topological group possess a uniform topology, and indeed one can define a notion of a left (resp. right) Cauchy sequence as follows: $\{x_n\}$ is a Cauchy sequence if for every neighborhood U of $e \in G$ there is an index $n_0 \in \mathbb{N}$ such that $m, n > n_0$ implies $x_m^{-1}x_n \in U$ (resp. $x_n x_m^{-1} \in U$).

Definition. 1) A topological vector space is called *sequentially complete* if every Cauchy sequence in it converges.

2) A subset $Y \subseteq X$ is called *sequencialy closed* if every Cauchy sequence $\{y_n\} \in Y$ converges to a point $y \in Y$.

The next example shows that we can have closed sets Y that are sequentially closed but not closed. This example also shows that if the topology is too strong (not first countable) then the notion of Cauchy sequence is not the "right notion".

Example. Let X be the real interval $[0,1]$ and let τ be the co-countable topology on X; that is, τ consists of X and $\emptyset$ together with all those subsets U of X whose complement U^C is a countable set. Let $A=[0,1)$, and consider $\overline{A}$. Now, $\{1\}\notin\tau$ because $X\setminus\{1\}=[0,1)$ is not countable. It follows that A is not closed. However, $\overline{A}$ is closed and contains A so $\overline{A}=[0,1]$. Since 1 is not an element of A, it must be a limit point of A. Suppose that $(a_n)_{n\in\mathbb{N}}$ is any sequence in A. Let $B=\{a_1,a_2,...\}$ and let $U=B^C$. Then $1\in U$ and since B is countable, it follows that U is an open neighborhood of 1 which contains no member of the sequence $(a_n)_{n\in\mathbb{N}}$. It follows that no sequence in A can converge to the limit point 1. This argument can be applied to show that A has no Cauchy sequences, so it is (trivially) sequentially closed but not closed.

Definition. 1) An embedding $i:V\hookrightarrow W$ is called a *strict embedding* if $i:V\hookrightarrow i(V)$ is an isomorphism of topological vector spaces.

2) A space V is called *complete* if for every strict embedding $\phi:V\to W$, the image $\phi(V)$ is closed.

Remark. * Equivalently, we can define that a space V is complete if any Cauchy net is convergent. By this definition it can be easily seen that any compete space X is also sequentially complete.

* In the category of first countable topological vector spaces, completeness is equivalent to sequentially completeness, and indeed the notion of Cauchy nets is equivalent to Cauchy sequence, and a set $Y\subseteq X$ is closed iff it is sequentially closed.

Exercise. Find a sequentially complete space which is not complete. **Hint**: See example.

Definition. 1) A space $\bar{V}$ will be called a *completion* of V if $\bar{V}$ is complete and there is a strict embedding $i:V\to\bar{V}$, where $i(V)$ is dense in $\bar{V}$.

2) A different definition can be made using a **universal property**: A (strict?) embedding $i:V\to\bar{V}$ is a *completion* of V if:

(a) $\bar{V}$ is complete.

(b) For every map $\psi:V\to W$ where W is complete, there is a unique map $\phi_W:\bar{V}\to W$, such that $\psi\equiv\phi_W\circ i$."

Exercise. (*) Show that these two definitions of completeness are equivalent.

This definition of completion, using the desired property saves us dealing with *Cauchy nets* or *filters*. However, one has to use them to show that such completion exists:

Exercise. 1) (*) Show that every linear topological Hausdorff space has a completion.

2) Show that in the category of first countable topological vector spaces $def1 \Longleftrightarrow def2 \Longleftrightarrow seq.comp.$

2.3. **Fréchet spaces. Reminder:** A *Banach space* is a normed space, which is complete with respect to its norm. A *Hilbert space* is a inner product space, which is complete with respect to its inner product.

Theorem. *(Hahn-Banach) Let V be a normed TVS, $W \subseteq V$ a linear subspace and $f : W \longrightarrow \mathbb{R}$ a continuous functional such that $|f(x)| < C \cdot \|x\|$, then there exists $\widetilde{f} : V \longrightarrow \mathbb{R}$ such that $\widetilde{f}|_W = f$ and $\left|\widetilde{f}(x)\right| < C \cdot \|x\|$.*

Exercise. Let $W \subseteq V$ be locally convex topological vector spaces, and set $V^\vee$ and $W^\vee$ to be the continuous duals of V and W respectively, and let $(*)$ denote the usual dual.

(a) Show that the restriction map $V^* \longrightarrow W^*$ is onto.

(b) Show that the restriction map $V^\vee \longrightarrow W^\vee$ is onto.

Every normed space is (Hausdorff and) locally convex, since the open balls in the space are convex, and they give a basis for the topology. We also know that every normed space is metric. However, metrizability doesn't force local convexity and vice versa.

Definition: A *Fréchet space* is a locally convex complete metrizable space.

Exercise. 1) Show that for a locally convex topological vector space V the following three conditions are equivalent, thus each implying that V is a Fréchet space.

(a) V is metrizable.

(b) V is first countable.

(c) There is a countable collection of semi-norms $\{n_i\}_{i \in \mathbb{N}}$ that defines the basis for the topology over V , i.e, $U_{i,\epsilon} = \{x \in V | n_i(x) < \epsilon\}$ is a basis for the topology.

2) Let V be a locally convex metrizable space. Prove V is complete (and it's a Fréchet space) iff it's sequentially complete.

Recall that a completion of a space using a norm is the quotient space of Cauchy sequences under the equivalence relation $\{x_n\} \sim \{y_n\} \iff \underset{n\to\infty}{lim}\|x_n - y_n\| = 0$. A completion of a space using a norm results in a Banach space. A completion of V using some semi-norm n will eliminate all the elements $\{x \in V \,|\, n(x) = 0\}$, and will define a norm on the quotient, again resulting in a Banach space. For example, the completion of the space of step functions on $\mathbb{R}$, with respect to the semi-norm $\|f\|_1 := \int_{\mathbb{R}} |f(x)|dx$ gives the Banach space $L^1(\mathbb{R})$.

Let V be a Fréchet space. In this case we have a sequence of semi-norms, n_i on V. We can order them by replacing n_i with $\underset{j\leq i}{max}\{n_j\}$. Denote V_i the completion of V with respect to n_i. If two **norms** n_i, n_j satisfy $\forall x \in V, n_i(x) \geq n_j(x)$, we get an inclusion (that is continuous) $V_i \hookrightarrow V_j$. A sequence of ascending norms $n_1 \leq n_2 \leq ...$ will thus give rise to a descending chain of completions $V_1 \hookleftarrow V_2 \hookleftarrow V_3 ....$ Our space V will be defined as the inverse limit $V = \underset{\longleftarrow}{lim} V_i$ which in this case has a very nice description: it is the intersection $V = \bigcap\limits_{i\in\mathbb{N}} V_i$ of these Banach spaces (with the subspace topology?). If n_i, n_j are semi-norms we only get a continuous map $V_i \hookrightarrow V_j$(every converging sequence is mapped to a converging sequence). In this case V will be the inverse limit $\underset{\longleftarrow}{lim} V_i$ where the topology on V is generated by all the sets of the form $\varphi_i^{-1}(U_i)$ where U_i is an open set in V_i and $\varphi_i : V = \underset{\longleftarrow}{lim} V_i \longrightarrow V_i$ is the natural map (it is part of the data of $\underset{\longleftarrow}{lim} V_i$).

Example. 1) Let $V := C^\infty(S^1)$ is a Fréchet space. Define the norms $\{n_i\}_{i\in\mathbb{N}}$ by $\|f\|_{n_i} := \underset{j\leq i}{max}\,\underset{x\in S^1}{sup}\{|f^{(j)}(x)|\}$. The completion with respect to n_k will be $V_k = C^k(S^1)$. This family satisfy $\forall x \in V, n_i(x) \geq n_j(x)$ so by the argument above we indeed have $C^\infty(S^1) = \bigcap\limits_{k\in\mathbb{N}} C^k(S^1)$.

2) $V = C^\infty(\mathbb{R})$ is a Fréchet space. Define $n_{K_i,n}$ by $\|f\|_{n_i} := \underset{j\leq i}{max}\,\underset{x\in K_i}{sup}\{|f^{(j)}(x)|\}$ where $K_i = [-i, i]$. Notice that this gives an ascending chain of seminorms so this defines a Fréchet space $V = \underset{\longleftarrow}{lim} V_i$. A similar argument can show $C^\infty(\mathbb{R}^n)$ is a Fréchet space, and actually also $C^\infty(M)$ for a manifold M. In these cases we'll take the supremum over all the possible derivatives.

Definition. *The direct limit* of an ascending sequence of vector spaces is the space $V_\infty := \bigcup\limits_{n\in\mathbb{N}} V_n$. This is not a Fréchet space, but a locally convex topological vector space. A convex subset $U \subseteq V_\infty$ will be open iff $U \bigcap V_n$ is open in V_n, for all n.

Every space $C^\infty(K)$ has the induced topology from $C^\infty(\mathbb{R})$. Taking the union of the ascending chain $C^\infty([-1, 1]) \subset C^\infty([-2, 2]) \subset ...$ will give all smooth functions with compact support $C_c^\infty(\mathbb{R}) = \underset{n}{lim}\, C^\infty([-n, n])$ as a direct limit. However, this is not a Fréchet space (it's a direct limit and not an inverse limit). A basic open set will be

$$U_{(\epsilon_n,k_n)} := \sum_{n\in\mathbb{N}} \{f \in C^\infty(\mathbb{R}) \,|\, Supp(f) \subseteq [-n,n], f^{(k_n)} < \epsilon_n\}$$

, where the Σ denotes the Minkowski sum, that is $A + B := \{a + b | a \in A, b \in B\}$.

Exercise. Show that $f_n \in C_c^\infty(\mathbb{R})$ converge to f with respect to the topology defined above if and only if it converges as was defined in the first lecture, i.e,

(a) There is a compact set $K \subseteq \mathbb{R}$ s.t. $supp(f) \cup supp(f_n) \subseteq K$.

(b) For every $k \in \mathbb{N}$ the derivatives $f_n^{(k)}(x)$ converge uniformly to $f^{(k)}(x)$.

Remark. Notice that the topology on $C_c^\infty(\mathbb{R})$ is complicated- it is a direct limit of an inverse limit of Banach spaces!

Exercise. Show that taking a convex hull instead of a Minkowski sum (i.e., defining $U_{(\epsilon_n,k_n)} := conv_{n\in\mathbb{N}}\{f \in C^\infty(\mathbb{R}) \,|\, Supp(f) \subseteq [-n,n], f^{(k_n)} < \epsilon_n\}$) will result in the same topology. This shows that $C_c^\infty(\mathbb{R})$ is a locally convex TVS (although by the definition as a direct limit of Fréchet spaces it is clearly a LCTVS).

Finally, Fréchet spaces have several more nice properties:

- Every surjective map $\phi : V_1 \to V_2$ between Fréchet spaces is an open map (it's actually enough that V_2 is a Fréchet space and V_1 is complete).
- Defining $K := ker\phi$, it can be shown that the quotient $V_1/_K$ is a Fréchet space, and factor ϕ to the composition $V_1 \to V_1/_K \to V_2$. The map $V_1/_K \to V_2$ will be an isomorphism.
- In addition, every closed map $\phi : V_1 \to V_2$ between Fréchet spaces can be similarly decomposed. First by showing $Im(\phi)$ is a Fréchet space, and then decomposing $V_1 \to Im(\phi) \to V_2$.

2.4. **Sequence spaces.** As an example for Fréchet spaces we'll analyze sequence spaces. **Reminder:** l^p is the space of all sequences $\{x_n\}_{n\in\mathbb{N}}$ over $\mathbb{R}$, such that $\sum\limits_{n=1}^{\infty} |x_n|^p < \infty$. It is a Banach space. For $p = 2$, it is also a Hilbert space.

Let $SW(\mathbb{N})$ be the space of all the sequences which decays to zero faster than any polynomial, i.e., $\forall n \in \mathbb{N}$, $\lim\limits_{i\longrightarrow\infty} x_i \cdot i^n = 0$. One norm over such sequences can be $||\{x_i\}||_n = \sup\limits_{i\in\mathbb{N}}\{|x_i \cdot i^n|\} = ||x_i \cdot i^n||_{l^\infty}$. In that norm we can easily see that every Cauchy sequence converges. Define the topology on $SW(\mathbb{N})$ by the family of norms $||\{x_i\}||_n$ and this defines a Fréchet space. Thus, this is an example for a Fréchet space which is not a Banach space.

QUESTION: How can we see every Cauchy sequence converges? Why isn't it a Banach space?

The dual space V^* will be $\left\{\{x_i\}_{i\in\mathbb{N}} \mid \exists n, c:\ x_i < c \cdot i^n\right\}$. This is a union of Banach spaces, as opposed to the intersection we had when defining the completion of a Fréchet space (we'll talk about the dual space more next lecture). Note that both V and V^* contain the subspace of all sequences with compact support - only finite number of non-zero elements.

QUESTION: Why is V^* the dual of V?

Actually, every separable space can be established as a sequence space. The elements of the space will correspond to infinite sequences. The elements in the countable dense subset of the space will correspond to the sequences with compact support.

Smooth functions on the unit circle, $C^\infty(S^1)$, correspond to sequences $\{x_i\}_{i\in\mathbb{N}}$ decaying faster than all polynomials. More precisely, we can view $f \in C^\infty(S^1)$ as a periodic function in $C^\infty_{periodic}(\mathbb{R})$ which can be written as $f(x) = \sum a_n \cdot e^{int}$. So we attach $f \longmapsto a_n$ and a_n decays faster then any polynomial.

Exercise. 1) Show that the Fourier transform $\mathcal{F} : C^\infty(S^1) \longrightarrow SW(\mathbb{Z})$ by $f \longmapsto a_n$ is an isomorphism of Fréchet spaces, that is, show that for any seminorm P_i of $SW(\mathbb{Z})$, there exists seminorm S_j of $C^\infty(S^1)$ and $C \in \mathbb{R}$ such that for any $f \in C^\infty(S^1)$, $\|\mathcal{F}(f)\|_{P_i} < C \cdot \|f\|_{S_j}$.

2) Define a Fréchet topology on $S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) | \lim_{x\to\pm\infty} f^{(n)}(x) \cdot x^k \longrightarrow 0 \forall k\}$.

3. Lecture 3- $C^{-\infty}(\mathbb{R}^n)$- topology and filtrations.

3.1. Topologies of the space of distributions.

Remark. Let $U \subseteq \mathbb{R}^n$ be an open set. Then we can define$C^{-\infty}(U) := (C_c^\infty(U))^*$.

Definition. 1) Let V be topological vector space. A subset $B \subseteq V$ is called *bounded* if for every open $U \subseteq V$ exists λ such that $B \subseteq \lambda \cdot U$. When the topology on V is given by a sequence of norms, B will bounded iff it is bounded with respect to every one of the norms.

2) Denote $V^* = \{f : V \to \mathbb{R} : f \text{ is linear and continuous}\}$. There are many topologies we can define on V^*, but we will consider only two topologies. V^* with the weak topology will be denoted V_W^*, and with and strong one V_S^*. Given $\epsilon > 0$ and $S \subseteq V$ denote $U_{\epsilon,S} = \{f \in V^* : \forall x \in S,\ f(x) < \epsilon\}$. The topology on V_w^* is induced by the basis:

$$\mathcal{B} := \{U_{\epsilon,S} : \epsilon > 0,\ |S| < \infty\},$$

while the topology on V_S^* is induced by the basis:

$$\mathcal{B} := \{U_{\epsilon,S} : \epsilon > 0,\ S \text{ is bounded}\}.$$

In particular, every open set in V_W^* is open in V_S^*.

By definition, a sequence $\{f_n\} \subseteq V^*$ converges to $f \in V^*$ iff for every $U_{\epsilon,S} \in \mathcal{B}$ there exists $N \in \mathbb{N}$ s.t. $(f_n - f) \in U_{\epsilon,S}$ for $n > N$. That is, $\forall x \in S,\ f_n(x) - f(x) < \epsilon$. Therefore $\{f_n\}$ converges to f under the weak topology iff it converges point-wise, and it converges under the strong topology iff it converges uniformly on every bounded set.

Example: Let $V = \mathbb{R}$. Let ψ be a bump function. Notice that $g_n(x) = \psi(x) + n$ converges pointwise to 0 (and hence also weakly). g_n doesn't converges uniformly to 0, but it **does** converges uniformly on bounded sets to 0 so it strongly converges to 0.

Assume V is a Fréchet space. Recall that we can define$V$ as a inverse limit of Banach spaces $V = \bigcap_{i \in \mathbb{N}} V_i$ where V_i is the completion of V with respect to an increasing sequence of semi-norms n_i. If we dualize the sequence $\{V_i\}$ we get an increasing sequence $V_1^* \subseteq V_2^* \subseteq ... \subseteq V_S^* = \varinjlim V_i^*$, and we get that V_S^* is a direct limit of Banach spaces (as a topological vector space).

Exercise. Consider the embedding $C_c^\infty(\mathbb{R}) \hookrightarrow C^{-\infty}(\mathbb{R})$, defined by $f \mapsto \xi_f$. Show that:

1) This embedding is dense with respect to the weak topology on $C^{-\infty}(\mathbb{R})$.

2) This embedding is dense with respect to the strong topology on $C^{-\infty}(\mathbb{R})$.

3) $C^{-\infty}(\mathbb{R})_w$ is not complete but it is sequentially complete.

4) $\overline{C^{-\infty}(\mathbb{R})_w} = C_c^{\infty}(\mathbb{R})^{\#}$- the dual space (as a linear space).

5) $\overline{C^{-\infty}(\mathbb{R})_S} = C^{-\infty}(\mathbb{R})_S$ - complete.

3.2. **Sheaf of distributions.**

Definition. Let $U_1 \subseteq U_2 \subseteq \mathbb{R}^n$ be open sets. Every function $f \in C_c^{\infty}(U_1)$ can be extended to a function $\tilde{f} \in C_c^{\infty}(U_2)$ by defining $\tilde{f}\mid_{U_2 \setminus U_1} \equiv 0$, hence we have an embedding $C_c^{\infty}(U_1) \hookrightarrow C_c^{\infty}(U_2)$. This embedding defines a restriction map $C^{-\infty}(U_2) \to C^{-\infty}(U_1)$, mapping $\xi \mapsto \xi\mid_{U_1}$, with $\xi\mid_{U_1}(f) := \xi(\tilde{f})$.

Remark. For an open $U \subset \mathbb{R}^n$, the topology on $C_c^{\infty}(U)$ is generally not the induced topology from $C_c^{\infty}(\mathbb{R}^n)$ under the embedding $C_c^{\infty}(U) \hookrightarrow C_c^{\infty}(\mathbb{R}^n)$. For every compact $K \subset U$, we have $C_K^{\infty}(U) \subset C_c^{\infty}(U)$. Here the topology on $C_K^{\infty}(U)$ is indeed the induced topology from $C_c^{\infty}(U)$.

We will prove next that with respect to the restriction of distributions defined above, the distributions form a sheaf.

Lemma: Let $f \in C_c^{\infty}(U)$, $U = \bigcup_{i \in I} U_i$. Then f can be written as a sum $f = \sum_{i \in I} f_i$ where $f_i \in C_c^{\infty}(U_i)$. Moreover, for every $x \in U$, the number of sets $|\{i \in I : f_i(x) \neq 0\}|$ will be finite.

Proof. We can assume that U_i are balls (otherwise, replace each U_i by the balls covering it). Denote $K := \mathit{supp}(f)$. It is a compact set covered by open balls, so there exists a finite sub-cover: $K \subseteq \bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} B(x_i, r_i)$. Since the cover is open and K is closed, there exists $\epsilon > 0$ such that $K \subseteq \bigcup_{i=1}^{n} B(x_i, r_i - \epsilon)$. Let ρ_i be smooth step functions satisfying $\rho_i|_{B(x_i, r_i - \varepsilon)} \equiv 1$, $\rho_i|_{B(x_i, r_i)^c} \equiv 0$. Since $\forall x \in K$, $\sum_{i=1}^{n} \rho_i(x) \neq 0$, we can define:

$$f_i = \begin{cases} \frac{\rho_i \cdot f}{\sum_{i=1}^{n} \rho_i} & x \in K \\ 0 & x \notin K \end{cases}$$

□

Theorem. *With respect to the restriction map defined above, the distributions form a sheaf, that is, given an open* $U \subseteq \mathbb{R}^n$*, and open cover* $U = \bigcup_{i \in I} U_i$*, we have:*

1) (Identity axiom) Let $\xi \in C^{-\infty}(U)$*. If* $\forall i$*,* $\xi|_{U_i} \equiv 0$*, then* $\xi|_U \equiv 0$*.*

2) (Glueability axiom) Given a collection $\{\xi_i\}_{i \in I}$*,* $\xi_i \in C^{-\infty}(U_i)$ *that agree on intersections (i.e.* $\forall i, j \in I$*,* $\xi_i|_{U_i \cap U_j} \equiv \xi_j|_{U_i \cap U_j}$*), there exists* $\xi \in C^{-\infty}(U)$*, satisfying* $\xi|_{U_i} \equiv \xi_i$ *for any* i*.*

Proof. 1) Given $f \in C_c^\infty(U)$ we need to show $\xi(f) = 0$. Indeed, by the lemma $f \equiv f_1 + ... f_n$, with $f_i \in C_c^\infty(U_i)$. Hence $\xi(f) = \xi(\sum_{i=1}^{n} f_i) = \sum_{i=1}^{n} \xi(f_i) = 0$.

2) We first use the fact that there exists a partition of unity, that is, $1 = \sum \lambda_i(x)$ where $supp(\lambda_i) \subseteq U_i$ and the sum is finite for any $x \in U$ an also that for any compact $K \subseteq U$ we have that $\lambda_i|_K \equiv 0$ for all but finitely many i's. Now fix some partition of unity $\{\lambda_i\}$ and let $\xi_i \in C^{-\infty}(U_i)$. Define $\xi(f) := \sum_{i \in I} \xi_i(\lambda_i f)$. Note that f is supported in some compact K so the sum is finite, so this is well defined. It is clear that ξ is linear. We need to prove that it is continuous, and that $\xi|_{U_i} = \xi_i$:

Let $f_n \to f \in C_c^\infty(U)$. Then also $\lambda_i \cdot f_n \to \lambda_i \cdot f$ as the multiplication $(f, g) \longmapsto f \cdot g$ is continuous. As $supp f_n \cup Supp f \subseteq K$ for some $K \subseteq U$, we have that $f\lambda_i \equiv 0$ for all but finitely many i's so we can write $\xi(f) := \sum_{i=1}^n \xi_i(\lambda_i f)$ and $\xi(f_n) := \sum_{i=1}^n \xi_i(\lambda_i f_n)$ for any n. By the continuity of ξ_i, $\xi_i(\lambda_i \cdot f_n) \to \xi_i(\lambda_i \cdot f)$ and therefore $\xi(f_n) = \sum_i \xi_i(\lambda_i \cdot f_n) \to \sum_i \xi_i(\lambda_i \cdot f) = \xi(f)$ so ξ continuous. Now let $f \in C_c^\infty(U_j)$, then

$$\xi(f) = \sum_i \xi_i(\lambda_i f) = \sum_i \xi_j(\lambda_i f) = \xi_j(\sum_i \lambda_i f) = \xi_j(f)$$

where the second equality follows from the fact that $\lambda_i f \in C_c^\infty(U_j \cap U_i)$ and $\xi_i|_{U_i \cap U_j} \equiv \xi_j|_{U_i \cap U_j}$.

There is also a second proof for the continuity of ξ is working with the open sets in the topology of $C_c^\infty(U)$: As ξ_i are continuous, they are bounded in some convex open set B_i of 0, so $\xi_i(B_i) < \epsilon$. Notice that $Conv(\cup B_i)$ is open in $\bigoplus_{i \in I} C_c^\infty(U_i)$ (where each B_i is an open set in $C_c^\infty(U_i)$ and hence a set in $\bigoplus_{i \in I} C_c^\infty(U_i)$), as $Conv(\cup B_i) \cap C_c^\infty(U_i) = B_i$. Notice that $\varphi(Conv(\cup B_i))$ is open. Now let $f \in \varphi(Conv(\cup B_i))$. We can write $f = \sum_{j_i=1}^n a_i f_i$ where $f_i \in B_{j_i}$ and $\sum a_i = 0$. Therefore $\xi(f) := \sum \xi_i(a_i f_i) < \sum a_i \cdot \epsilon = \epsilon$ and ξ is bounded on B. □

3.3. **Filtration on a space of distributions.**

Exercise. $V := \overline{C_c^\infty(U)} = \{f \in C_c^\infty(\mathbb{R}^n) : \forall x \notin U,\ \forall \text{ differential operator } L,\ Lf(x) = 0\}$.

Consider $U = \mathbb{R}^n \setminus \mathbb{R}^k$. We wish to describe the space of distributions supported in $\mathbb{R}^k$, denoted $C_{\mathbb{R}^k}^{-\infty}(\mathbb{R}^n)$. Notice that:

$$C_{\mathbb{R}^k}^{-\infty}(\mathbb{R}^n) = \{\xi \in C^{-\infty}(\mathbb{R}^n) | \forall f \in C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^k) \text{ it holds that } \xi(f) = 0\}$$

and by continuity this equals

$$= \{\xi \in C^{-\infty}(\mathbb{R}^n) | \forall f \in \overline{C_c^\infty(\mathbb{R}^n/\mathbb{R}^k)} \text{ it holds that } \xi(f) = 0\} = \{\xi | \, \xi|_V = 0\}$$

Notice that we can define a natural descending filtration on V by:

$$V \subseteq V_m = \{f \in C_c^\infty(\mathbb{R}^n) | \forall i \in \mathbb{N}^{n-k} \text{ where } |i| \leq m \text{ it holds that } \frac{\partial^i f}{(\partial x)^i}|_{\mathbb{R}^k} = 0\}$$

We see immediately that $f \in V_m(C_c^\infty(\mathbb{R}^n))$ implies $f \in V_{m-1}(C_c^\infty(\mathbb{R}^n))$, hence this is a descending chain. Accordingly, we can define a ascending filtration on $C_{\mathbb{R}^k}^{-\infty}(\mathbb{R}^n)$ by:

$$F_m(C_{R^k}^{-\infty}(\mathbb{R}^n)) = V_m^* = \{\xi \in C_{R^k}^{-\infty}(\mathbb{R}^n) : \xi|_{V_m} = 0\} \subseteq C_{R^k}^{-\infty}(\mathbb{R}^n).$$

Exercise. 1) $\cap V_m = V = \overline{C_c^\infty(\mathbb{R}^n \setminus \mathbb{R}^k)}$.

2) $\cup F_m \neq C_{R^k}^{-\infty}(\mathbb{R}^n)$.

3) Let $U \subseteq \mathbb{R}^n$ be open and $\overline{U}$ compact. Show that for every $\xi \in C_{\mathbb{R}^k}^\infty(\mathbb{R}^n)^*$ there exists $\xi' \in F_m$ such that $\xi|_U = \xi'|_U$, thus $\bigcup_{i=0}^\infty F_i$ covers $C_{\mathbb{R}^k}^\infty(\mathbb{R}^n)^*$ locally.

4) Consider a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ that fixes $\mathbb{R}^k$. Show that changing coordinates using φ for $\xi \in F_i$ we get a distribution in F_i (so F_i is preserved under change of coordinates: $\varphi^*(F_i) = F_i$, meaning: $\forall \xi \in F_i,\ \xi(\varphi(f)) \in F_i$).

Theorem. $F_m \simeq \oplus_{i \in \mathbb{N}^{n-k}, |i| \leq m} \frac{\partial^i (C^{-\infty}(\mathbb{R}^k))}{(\partial x)^i}$ *as vector spaces.*

Proof. 1) We will prove for $m = 0$. Let $\xi \in C^{-\infty}(\mathbb{R}^k)$, we can assign $\phi : \xi \longmapsto \widetilde{\xi} \in F_0$ by $\widetilde{\xi}(f) = f|_{\mathbb{R}^k}$. Notice that $\widetilde{\xi}(f) = 0$ for any $f \in F_0$ so it is well defined. It is clear that it is injective as if $\widetilde{\xi}(f) = 0$ for all $f \in C_c^\infty(\mathbb{R}^n)$ then $\xi(f|_{\mathbb{R}^k}) = 0$ for any f but any map $g \in C_c^\infty(\mathbb{R}^k)$ can be extended to $\widetilde{g}$ such that $\widetilde{g}|_{\mathbb{R}^k} = g$. It is left to prove surjectivity and we are done. Let $\eta \in F_0$. Assign $\eta \longmapsto \widetilde{\eta} \in C^{-\infty}(\mathbb{R}^k)$ by $\widetilde{\eta}(f) := \eta(\widetilde{f})$ where $\widetilde{f}$ satisfy $\widetilde{f}|_{\mathbb{R}^k} = f$.

Notice that $\widetilde{\eta}$ is well defined as if $\widetilde{f}, \widetilde{g}$ satisfy $\widetilde{f}|_{\mathbb{R}^k} = \widetilde{g}|_{\mathbb{R}^k} = f$ then $\eta(\widetilde{g}) = \eta(\widetilde{f})$. Also $\widetilde{\eta}$ is continuous as if $f_n \longrightarrow f$ then we can choose lifts such that $\widetilde{f_n} \longrightarrow \widetilde{f}$ and as η is continuous it follows. We got that $\phi(\widetilde{\eta}) = \eta$ so ϕ is surjective and we are done. Now for $m > 0$ this is a generalization to the exercise that any distribution supported on 0 is a combination of derivatives of δ, and it will be proved in lecture 5. $\square$

We can define $G_m = \oplus_{i \in \mathbb{N}^{n-k}, |i|=m} \frac{\partial^i (C^{-\infty}(\mathbb{R}^k))}{(\partial x)^i}$ and hence $G_m \simeq F_m / F_{m-1}$.

Exercise. Show that this decomposition is not invariant under change of coordinates, that is $\varphi(G_m) \neq G_m$, $\varphi(G_{(i)}) \neq G_{(i)}$ where (i) is a multi-index.

4. Lecture 4 - P adic numbers and L spaces

We want to find integer solution for an equation of the form of an integer valued polynomial $p(X) = 0$. If there exists such a solution X, then it must satisfies the same equation modulo some prime p and also modulo p^n for any $1 < n \in \mathbb{N}$. Therefore we want to define some "creature" denoted $\mathbb{Z}_p$, such that having a solution in $\mathbb{Z}_p$ is the same as having a solution $modp^n$ for any n. For this we will need to define the *p-adic numbers*.

4.1. **Defining p-adic numbers.**

Definition. 1) A *topological field* is a field F, together with a topology, such that addition, multiplication and the multiplicative and additive inverses are continuous operations.

2) Given a field F, an *absolute value* is a function $|\ | : F \to \mathbb{R}^+$ that satisfies:

* The triangle inequality : $|x + y| \leq |x| + |y|$.

*$|x||y| = |xy|$.

*$|x| = 0 \Leftrightarrow x = 0$.

For topological field we demand the absolute value to be continuous map. Notice that every absolute value satisfies $|1| = 1$ (as $|1| = |1| \cdot |1|$, and $|1| \neq 0$).

Example. 1) The trivial absolute value, defined by: $|x|_0 := \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$

2) The standard absolute value on $\mathbb{R}$, which we'll denote $|\ |_\infty$.

Now if we want to solve the equation $f(X) = 0$ modulo p^n we want p^n to be zero. Therefore, when we define the p-adic norm we want $\|p^n\|$ to be "very small" as n grows.

Definition. For any given integer a we define the $ord_p a$ to be the highest power m of p such that $p^m|a$. For $x = a/b \in Q$ we define $ord_p x = ord_p a - ord_p b$. Define the $p - adic\, norm$ by $\|x\|_p = 0$ if $x = 0$ and $\|x\|_p = 1/p^{ord_p x}$ otherwise.

Proposition. *The map* $\|\ \|_p$ *defined above gives an absolute value on* $\mathbb{Q}$.

Proof. $\|a/b\|_p = 0$ then either $a/b = 0$ or $ord_p a = \infty$ and therefore $a/b = 0$. Note that:

$$\left\|\frac{ac}{bd}\right\|_p = \frac{1}{p^{ord_p(ac)-ord_p(bd)}} = \frac{1}{p^{ord_p(a)-ord_p(b)}} \cdot \frac{1}{p^{ord_p(c)-ord_p(d)}} = \left\|\frac{a}{b}\right\|_p \cdot \left\|\frac{c}{d}\right\|_p$$

and also that triangle inequality holds:

$$\left\|\frac{a}{b}+\frac{c}{d}\right\|_p = \left\|\frac{ad+bc}{bd}\right\|_p = 1/(p^{ord_p(ad+bc)-ord_p(bd)}) \leq 1/(p^{min(ord_p(ad),rd_p(bc))-ord_p(b)-ord_p(d)}) =$$

$$1/(p^{min(ord_p(a)+ord_p(d),ord_p(b)+ord_p(c))-ord_p(b)-ord_p(d)}) = 1/(p^{min(ord_p(a)-ord_p(b),ord_p(c)-ord_p(d))}) =$$

$$1/(p^{min(ord_p(x),ord_p(y))}) = max(\left\|\frac{a}{b}\right\|_p, \left\|\frac{c}{d}\right\|_p) \leq \left\|\frac{a}{b}\right\|_p + \left\|\frac{c}{d}\right\|_p.$$

□

Definition. A norm is called *non archimedean* if $\|x+y\| \leq max(\|x\|, \|y\|)$ always holds. In particular, if $\|x\| \neq \|y\| \Rightarrow \|x+y\| = max(|\,\|x\|, \|y\|)$. Note that by the last proposition,$\| \|_p$ is non archimedean.

2) Two norms $|\ |, |\ |'$ on F are called *equivalent* (denoted $|\ | \sim |\ |'$) if for any $\{a_n\} \in \mathbb{Q}$, a_n is a Cauchy sequence with respect to $|\ |$ iff it is a Cauchy sequence with respect to $|\ |'$.

Theorem. *(Ostrowski Theorem) Every non-trivial norm* $\| \|$*on* $\mathbb{Q}$ *is equivalent to* $\| \|_p$ *for some* p*, or the usual norm on* $\mathbb{Q}$ *induced from* $\mathbb{R}$*, denoted* $\| \|_\infty$.

Proof. Case (i): There exists $n \in \mathbb{N}$ such that $\|n\| > 1$. Let n_0 be the least such n. So there exists $0 < \alpha < 1$ $\|n_0\| = n_0^\alpha$. We write each n in the base of n_0, that is we choose $0 \leq \{a_i\} < n$ and $\{a_i\} \in \mathbb{N}$ such that $n = a_0 + a_1 n_0 + ... a_k n_0^k$. Note that:

$$\|n\| = \left\|a_0 + a_1 n_0 + ... a_k n_0^k\right\| \leq \|a_0\| + \|a_1\| \cdot n_0^\alpha + ... + \|a_k\| \cdot n_0^{\alpha k}$$

By the choice of n_0 we have that $\|a_i\| \leq 1$ so

$$\|n\| \leq \sum_i n_0^{i\alpha} = n_0^{k\alpha}(\sum(1 + n_0^{-\alpha} + ... n_0^{-k\alpha}) \leq n^\alpha \cdot \left(\sum_{t=1}^{\infty} \left(n_0^{-\alpha}\right)^t\right) = n^\alpha \cdot C$$

since $n \geq n_0^k$. Therefore $\|n\| \leq n^\alpha \cdot C$ and the constant C doesn't depend on n. By choosing large enough N we can show that $\left\|n^N\right\| \leq n^{N\alpha} \cdot C$ so $\|n\| \leq n^\alpha \cdot C^{1/N}$ for any N. This implies that $\|n\| \leq n^\alpha$

Now we get the inequality in the other direction also: if n is written in the base of n_0 as before, we have that $n_0^{k+1} > n \geq n_0^k$. Also

$$\left\|n_0^{k+1}\right\| = \left\|n + n_0^{k+1} - n\right\| \leq \|n\| + \left\|n_0^{k+1} - n\right\|,$$

so using $\|n\| \leq n^\alpha$and $n \geq n_0^k$ we get:

$$\|n\| \geq \left\|n_0^{k+1}\right\| - \left\|n_0^{k+1} - n\right\| \geq n_0^{\alpha(k+1)} - \left(n_0^{k+1} - n\right)^\alpha \geq n_0^{\alpha(k+1)} - \left(n_0^{k+1} - n_0^k\right)^\alpha$$

$$= n_0^{\alpha(k+1)}(1 - \left(1 - \frac{1}{n_0}\right)^\alpha \geq n^\alpha C'(n_0, \alpha).$$

.

Again, $\|n\| \geq n^\alpha$ so $\|n\| = n^\alpha$. This defines the norm uniquely on all $\mathbb{Q}$ as $\|ab\| = \|a\| \cdot \|b\|$ so taking $a = m, b = n/m$ we get $\|n/m\| = (n/m)^\alpha$. By writing Cauchy sequences we see that $\| \|^\alpha$is equivalent to $\| \|_\infty$.

case ii) For any n $\|n\| \leq 1$:

Let $n_0$be the least n such that $\|n\| < 1$ (otherwise $\|n\| = 1$ for any $n \neq 0$). n_0 must be a prime since if $n_0 = n_1 \cdot n_2$ then the norm of n_1 or n_2 must be smaller than 1 and we get a contradiction to the minimality of n_0. Denote $p = n_0$. We claim that $\|q\| = 1$ if $q \neq p$ prime:

Suppose $\|q\| < 1$, so for large N we have $\left\|q^N\right\| < 1/2$. Also, for large M we have $\left\|P^M\right\| < 1/2$. since $gcd(p^M, q^N) = 1$ then there exists m, n such that $mp^M + nq^N = 1$ but then:

$$1 = \|1\| = \left\|mp^M + nq^N\right\| \leq \left\|mp^M\right\| + \left\|nq^N\right\| = \|m\| \cdot \left\|p^M\right\| + \|n\| \left\|q^M\right\| < 1/2 + 1/2 < 1$$

So $\|q\| = 1$. Now let $a = p_1^{b_1} \cdot ... \cdot p_r^{b_r}$. If we denote $\|p\| = \rho$ then we get that$\|a\| = \|p\|^{ord_p(a)} = \rho^{ord_p(a)}$. This defines the norm uniquely on $\mathbb{Q}$. It is an easy exercise to show that this norm is equivalent to $\| \|_p$. □

Proposition. *Show that for any 2 norms on a field* $\| \|_1, \| \|_2$*are equivalent iff* $\| \|_1 = \| \|_2^\alpha$.

Proof. Assume $\| \|_1 = \| \|_2^{\alpha}$. Then it is clear that $\|a_n\| \longrightarrow 0$ iff $\|a_n\|^{\alpha} \longrightarrow 0$. Now let $\| \|_1, \| \|_2$ be equivalent norms . We divide to cases, according to Ostrowski theorem:

Case1:$\| \|_1 = \| \|^{\alpha}$ and $\| \|_2 = \| \|^{\beta}$ for $0 < \alpha, \beta < 1$.

Case2: $\| \|_1 \sim \| \|_p, \| \|_2 \sim \| \|_q$. If $q \neq p$ then the norms are not equivalent as $\{p^k\}$ is a Cauchy series in $\| \|_1$ but not in $\| \|_q$. Therefore by the proof of the last theorem, $\|a\|_1 = \|p\|^{ord_p(a)} = \rho_1^{ord_p(a)}$ and $\|a\|_2 = \|p\|^{ord_p(a)} = \rho_2^{ord_p(a)} = \rho_1^{\beta ord_p(a)}$ for any $a \in \mathbb{Z}$ and therefore $\| \|_2 = \| \|_1^{\beta}$.

Case 3: $\| \|_1 \sim \| \|_p, \| \|_2 \sim \| \|_{\infty}$(or the opposite) . Then $\{p^k\}$ is a Cauchy series in $\| \|_1$ but not in $\| \|_2$. □

Proposition. *Prove that addition, multiplication, and inverse is continuous for any norm on a field F.*

Proof. Let $\epsilon > 0$ and $x, y \in F$. Let x', y' be such that $\|x' - x\| < \epsilon/2$,$\|y' - y\| < \epsilon/2$. Then $\|(x + y) - (x' + y')\| \leq \|x' - x\| + \|y' - y\| < \epsilon$. For multiplicity inverse: Let x and $\epsilon > 0$, we have

$$\|x - x'\| < \delta = min(\epsilon \cdot \|x\|^2 /2, 1/2)$$

then $\|x - x' + x'\| \leq \|x - x'\| + \|x'\| < \delta + \|x'\|$ and the same for $\|x' - x + x\| \leq \|x - x'\| + \|x\| < \delta + \|x\|$so $|\|x\| - \|x'\|| < \delta$.

$$\|1/x' - 1/x\| = \left\| \frac{x' - x}{xx'} \right\| = \|x - x'\| / \|xx'\| =$$

$$\|x - x'\| / \|x\| \|x'\| < \delta \cdot \frac{1}{\|x\|^2 (1 - \delta)} < \epsilon \|x\|^2 /2 \cdot \frac{2}{\|x\|^2} = \epsilon.$$

The same idea for additive inverse, and multiplication. □

There are several nice properties of a non-archimedean norm:

1) every triangle (x, y, z) is isosceles! ("shve shokaim").

2) every open ball of radius r with center x has all of its points as a center as well.

Proof: Let $y \in B(x, r)$, and $z \in B(y, r)$, then $\|z - x\| = \|z - y + y - x\| \leq max(\|z - y\|, \|y - x\|) \leq r$ so $z \in B(x, r)$ so $B(y, r) \subseteq B(x, r)$. The other way, let $w \in B(x, r)$, then $\|w - y\| = \|w - x + x - y\| \leq max(\|w - x\|, \|y - x\|) \leq r$. So $B(y, r) = B(x, r)$.

3) Every Ball $B_r(x)$ is open and closed.

4) If 2 p-adic balls are not distinct, then one of them contains the other.

Proof: Let be $B(x,r_1)$ and $B(y,r_2)$ such that $B(x,r_1)\cap B(y,r_2)\neq\emptyset$ then there exists z such that $\|z-x\|<r_1$ and $\|z-y\|<r_2$. Lets assume $r_2\geq r_1$ Let $w\in B(x,r_1)$ then

$$\|w-y\|=\|w-x+x-z+z-y\|\leq max(\|w-x\|,\|x-z\|,\|z-y\|)\leq r_2.$$

Now we can define the p-adic numbers.

Definition. Let p be a prime number. The *field of p-adic numbers*, denoted $\mathbb{Q}_p$, is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value. The completion is defined just as we did in the case of the archimedean norm on $\mathbb{Q}$- by equivalence classes of Cauchy sequences. Therefore, any element $a\in\mathbb{Q}_p$ is represented by a Cauchy sequence $\{a_n\}\in\mathbb{Q}$ with respect to $\|\,\|_p$. We say that $a\sim b$ if $\{a_n\}\sim\{b_n\}$ if $\|a_n-b_n\|_p\longrightarrow 0$ and we define the norm of $a\in\mathbb{Q}_p$ by $lim_{i\to\infty}\|a_i\|_p$ (it exists by the following proposition).

Remark. Notice that just like $\mathbb{R}$, this completion is not algebraically closed. Try to find an equation in $\mathbb{Q}_p$ when the solution is not in $\mathbb{Q}_p$.

Proposition. *If $\{a_i\}$ is a Cauchy series in $\mathbb{Q}$ with respect to $\|\,\|_p$, then $lim_{i\to\infty}\|a_i\|_p$ exists.*

Remark. If $\{a_i\}$ equivalent to $\{0\}$ then by definition it exists. Else, for every $\epsilon>0$ there is a sub-sequence a_{i_k}such that $\|a_{i_k}\|_p>\epsilon$. We take N large enough such that $\|a_i-a_{i'}\|_p<\epsilon$ for every $i,i'>N$, by the Cauchy property. Then $\|a_i-a_{i_k}\|_p<\epsilon$, so $\|a_{i_k}-a_i\|_p<\|a_{i_k}\|_p$ so by the property that every triangle is isosceles, we have that $\|a_{i_k}\|_p=\|a_{i_k}-(a_{i_k}-a_i)\|_p=\|a_i\|_p$. So there exists N such that $\|a_i\|_p$ is constant for $i>N$.

Theorem. *$\mathbb{Q}_p$ is complete.*

Proof. Let $\{a_j\}\in\mathbb{Q}_p$ be a sequence of equivalence classes with$\{a_{ji}\}$ their representatives as Cauchy sequences in $\mathbb{Q}$. Assume that $\{a_j\}$ is Cauchy, i.e, there exists M such that for any $j,j'>M$:

$$\|\{a_j-a_{j'}\}\|:=\lim_i\|a_{ji}-a_{j'i}\|<\epsilon.$$

This means that there is $N_{j,j'}$ such that for $i>N_{j,j'}$: $\|a_{ji}-a_{j'i}\|<\epsilon$. In particular, for any j, there exists N_j such that for any $i,i'\geq N_j$: $\|a_{ji}-a_{ji'}\|<p^{-j}$. We claim that $\{b\}=\{a_{kN_k}\}$ is the limit of $\{a_j\}$. Notice that:

$$\|\{b-a_j\}\|=\lim_k\|\{a_{kN_K}-a_{jk}\}\|=\lim_k\left\|a_{kN_k}-a_{kN_{j,k}}+a_{kN_{j,k}}-a_{jN_{jk}}+a_{jN_{jk}}-a_{jk}\right\|.$$

For large enough k we have, for any $j > M$:

$\left\|a_{kN_k} - a_{kN_{j,k}} + a_{kN_{j,k}} - a_{jN_{jk}} + a_{jN_{jk}} - a_{jk}\right\| \leq max(\left\|a_{kN_k} - a_{kN_{jk}}\right\|, \left\|a_{kN_{jk}} - a_{jN_{jk}}\right\|, \left\|a_{jN_{jk}} - a_{jk}\right\|) -$

So $\{b\}$ is indeed the limit. □

Theorem. *Every equivalence class $a \in \mathbb{Q}_p$ for which $\|a\|_p \leq 1$ has exactly one representative Cauchy sequence of the form $\{a_i\}$ for which:*

1) $0 \leq a_i < p^i$ for $i = 1, 2, ...$

2) $a_i \equiv a_{i+1}(mod(p^i))$ for $i = 1, 2, ...$

Proof. At first we prove the uniqueness: If $\{a'_i\}$ is a different sequence satisfying (1) and (2) and if there exists i_0 such that $a_{i_0} \neq a'_{i_0}$ then $a_i \neq a'_i(mod(p^{i_0}))$ for every $i > i_0$. Therefore $\|a_i - a'_i\| > 1/p^{i_0}$ so $\{a'_i\}, \{a_i\}$ are not equivalent. Now we prove existence: Suppose we have a Cauchy sequence $\{b_i\} \in Q_p$, we want to find an equivalent sequence $\{a_i\}$ with the above property. We use the following lemma: □

Lemma. *If $x \in \mathbb{Q}$ and $\|x\|_p \leq 1$, then for any i there exists an integer $\alpha \in \mathbb{Z}$ such that $\|\alpha - x\|_p \leq p^{-i}$. The integer α can be chosen in the set $\{0, 1, 2, ...p^i - 1\}$.*

Proof. Let $x = a/b$ written in the form where $(gcd(a, b) = 1)$. Since $\|x\|_p \leq 1$ it follows that p does not divide b and therefore b and $p^i$are relatively prime. Then we can find $m, n \in \mathbb{Z}$ such that $bm + np^i = 1$. The intuition is that bm is close to 1 up to a small p-adic length so it is a good approximation to 1 so am is a good approximation to a/b. So we pick $\alpha = am$ and get:

$$\|\alpha - x\| = \|am - a/b\| = \|a/b\| \cdot \|bm - 1\| \leq \|bm - 1\| = \left\|np^i\right\| \leq 1/p^i$$

Note that we can add multiples of p^i to α and still have

$$\left\|\alpha - k \cdot p^i - x\right\| \leq max(1/p^i, 1/p^i) \leq 1/p^i.$$

Therefore we can assume that $\alpha \in \{0, ...p^i - 1\}$. □

Now back to the proof:

Proof. Back to $\{b_i\}$. Let N_j be the number such that for every $i, i' > N_j$ we have $\|b_i - b_{i'}\| < p^{-j}$, and we can choose N_j to be strictly increasing with j, and $N_j > j$.

Observe that $\|b_i\| \leq 1$ if $i > N_1$. Indeed, for all $i' > N_1$ we have that $\|b_i - b_{i'}\| < 1/p$, $\|b_i\| \leq max(\|b_{i'}\|, \|b_i - b_{i'}\|)$ and for $i' \to \infty$ we have that $\|b_{i'}\| \to \|a\|_p \leq 1$.

Now we use the lemma and get a sequence $\{a_j\}$ when $0 \leq a_j < p^j$ such that $\left\|a_j - b_{N_j}\right\| < p^{-j}$. We claim that$\{a_j\}$ is equivalent to $\{b_i\}$, and satisfies the conditions of the theorem. It is indeed satisfies the conditions as:

$$\|a_{j+1} - a_j\| = \left\|a_{j+1} - b_{N_{j+1}} + b_{N_{j+1}} - b_{N_j} - (a_j - b_{N_j})\right\|$$

$$\leq max(\left\|a_{j+1} - b_{N_{j+1}}\right\|, \left\|b_{N_{j+1}} - b_{N_j}\right\|, \left\|a_j - b_{N_j}\right\|) \leq p^{-j}$$

So $a_{j+1} - a_j$ has at least p^jas a common divisor as required.

Furthermore, for any j and any $i > N_j$:

$$\|a_i - b_i\| = \left\|a_i - a_j + a_j - b_{N_j} - (b_i - b_{N_j})\right\| \leq max(\|a_i - a_j\|, \left\|a_j - b_{N_j}\right\|, \left\|b_i - b_{N_j}\right\|) \leq p^{-j}.$$

So $\{a_i\} \backsim \{b_i\}$. □

Now, if we have some $\{a\} \in \mathbb{Q}_p$ with $\|a\| \geq 1$ then there exists some m such that $\|a \cdot p^m\| \leq 1$ and we have numbers with negative powers. Therefore we can present the p-adic numbers as:

$$\mathbb{Q}_p := \{\sum_{i=-k}^{\infty} a_i \cdot p^i, \text{ where } a_i \in \{0...p^i - 1\}\}.$$

We define the *ring of integers* , denoted $\mathbb{Z}_p$ as $\mathbb{Z}_p := \{x \in \mathbb{Q}_p | \, \|x\|_p \leq 1\}$ or equivalently $\mathbb{Z}_p := \{\sum_{i=0}^{\infty} a_i \cdot p^i, \text{ where } a_i \in \{0...p^i - 1\}\}$ or equivalently $\mathbb{Z}_p := \overline{\mathbb{Z}}_{\| \|_p}$- the closure of $\mathbb{Z}$ with respect to the p-adic norm. Notice that $\mathbb{Z}_p$ is indeed a ring and that the only invertible elements are $x \in \mathbb{Z}_p$ with $\|x\|_p = 1$.

4.2. **p-adic expansions.** We want to write the p-adic expansions of elements q in $\mathbb{Q}$. If $q \in \mathbb{N}$, that's just writing its p-base expansion. For example, $(126)_5 =$ "...002001." Let $x := \frac{m}{n}$ be some rational number, with $(n, m) = 1$. It is enough to describe the expansion when $p \nmid m$ (that is, when $x \in \mathbb{Z}_p \cap \mathbb{Q}$) as otherwise we can multiply x by p^k for some k, calculate the expansion, and move the point k places to the left.

We can't take remainder of x modulo p, as with integers. Instead, we can calculate the fraction $x = \frac{m}{n}$ in $\mathbb{F}_{p^k}$ for $k \in \mathbb{N}$. Thus, the expansion of x in $\mathbb{Q}_p$ is calculated inductively:

- Write the digit $x_0 := [\frac{m}{n}] \in \mathbb{F}_p$.
- The nominator of the difference $\frac{m}{n} - x_0 = \frac{m-n\cdot x_0}{n}$ is divisible by p. Redefine our fraction to be $x := \frac{1}{p} \cdot (\frac{m}{n} - x_0)$, and continue inductively.

Example. Calculate $\frac{1}{2} \in \mathbb{Q}_7$. We start by solving the equation $2x_0 = 1(mod 7)$. The answer is $x_0 = 4$. In the second case we calculate $\frac{1}{7}(\frac{1}{2}-4) = x_1$. So $2\cdot(7x_1+4) = 1(mod 49)$. Therefore $x_1 = 3$. We continue by induction and get the required expansion.

Every ball in $\mathbb{Q}_p$ is a disjoint union of p balls. For $p = 2$, the ball $\mathbb{Z}_2 = B_c(0,1) = B_o(0,2)$ consists of numbers with no digits to the right of the point. It's a disjoint union of two balls, B_0 and B_1 - where each B_i consists of all numbers ending with the digit $'i'$. Similarly, $B_0 = B_{00} \bigcup B_{01}$, $B_1 = B_{10} \bigcup B_{11}$, where the elements in B_{ij} end with the digits $'ij'$. And so on.

This recursive structure implies p-adic integers are homeomorphic to the Cantor set.

Exercise. Show $\mathbb{Z}_p \cong$ *Cantor set* as topological spaces, where the Cantor set has the topology induced by the real numbers. The exercise proves $\mathbb{Z}_p$ is a compact set.

4.3. Inverse limits.

Definition. Let $A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow ...$ be a sequence of Abelian groups $\{A_i\}$ together with a set of homomorphisms $\{f_{ij} : A_j \to A_i \mid j > i\}$, such that $f_{ik} = f_{ij} \circ f_{jk}$, $\forall i \leq j \leq k$. An *inverse limit* of a sequence of Abelian groups is defined by:

$$\underleftarrow{lim} A_i = \{\overrightarrow{a} \in \prod_{i\in\mathbb{N}} A_i : a_i = f_{ij}(a_j), \forall i \leq j \in \mathbb{N}\}$$

Exercise. 1) Take $A_i := \mathbb{Z}/p^i\mathbb{Z}$, and f_{ij} to be the projection $\mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$. Prove that $\underleftarrow{lim}\mathbb{Z}/p^n\mathbb{Z} \simeq \mathbb{Z}_p$ as a topological ring.

2)$\mathbb{Q}_p$ is the localization by p of $\mathbb{Z}_p$ and $\mathbb{Q}_p = p^{-1}\mathbb{Z}_p = \{p^{-k}a | a \in \mathbb{Z}_p\} \simeq \underleftarrow{lim}\mathbb{Q}/p^n\mathbb{Z}$, again, as topological rings.

2) Prove that $\mathbb{Q}_p \simeq$ Cantor set - $\{1\}$.

3) Prove that $\mathbb{Q}_p^n \cong \mathbb{Q}_p$.

4) Let $U \subset \mathbb{Q}_p^n$ be some open set. Show that either U is homeomorphic to the Cantor set, or to Cantor set-$\{1\}$.

4.4. **Haar measure and local fields.** Let X be a topological space. Let $C_c(X)$ be the space of continuous functions with compact support on X and consider $C_c(X)^*$- the space of smooth measures.

Theorem. ***(Haar):*** *Let G be a locally compact topological group. Then:*

1) There exists a measure μ on X such that $\mu(U) = \mu(gU)$ for any measurable set (Or equivalently, there exists $\phi \in C_c(X)^*$ such that for any $g \in G$, $\phi(f) = \phi(f_g)$ where $f_g(x) = f(g^{-1} \cdot x)$).

2) This measure is unique up to a scalar.

Exercise. 1) Prove Haar theorem for $(\mathbb{Q}_p, +)$.

2) We can define another invariant measure $\mu_a(B) = \mu(aB)$ for any $a \in \mathbb{Q}_p$. Show that $\mu_a = |a| \cdot \mu$.

Definition. A *local field* is a topological field that is not discrete and locally compact.

Theorem. *Any local field F is isomorphic (as topological field) to one of the following:*

** Archimedean fields- $\mathbb{R}$ or $\mathbb{C}$.*

** Finite extensions of $\mathbb{Q}_p$.*

** Finite extensions of the formal Laurent series: - $\mathbb{F}_q((t)) = \{\sum_{i=-k}^{\infty} a_i t^i\}$ where $\mathbb{F}_q$ is a finite field (so q may be some power of p).*

Proof. The main points of the proof are as follows:

(1) Define the measure on F^+ using Haar Theorem. We can define absolute value, up to scalar multiplication, that is, there exists $\alpha(a)$ such that, $\mu_a = \alpha(a)\mu \to |a| \equiv \alpha(a)$.

(2) Prove that every local field has a norm that defines its topology, which defined as a scalar multiplication of Haar measure.

(3) Prove that every compact metric space is complete.

(4) Every local field of char 0 includes $\mathbb{Q}$ and its completion. This means that F contains $\mathbb{R}$ if it is archimedean, and $\mathbb{Q}_p$ if it is non-archimedean.

(5) Show that if F is characteristic 0, then F must be a finite extension of $\mathbb{R}$ or $\mathbb{Q}_p$, otherwise (non algebraic extension) it will not be compact. Show that any such finite extension is indeed a local field.

(6) For $char(F) \neq 0$ show that F contains a transcendental element, name it t, and show that it contains $\mathbb{F}_q((t))$. Show that F is a finite extension of $\mathbb{F}_q((t))$. $\square$

4.5. **Some basic properties of l-spaces.**

Definition. An $l-space$ X is an Hausdorff, locally compact an totally disconnected topological space.

Exercise. 1) This definition is equivalent to having a basis of open compact subsets (and being Hausdorff).

2) Any non-archimedean local field is an l-space.

3) Finite products, and open/closed subsets of an l-space is an l-space. Note that any subset of a totally disconnected topological space is totally disconnected.

Definition. A space is called *countable at* ∞ if $X = \cup_{n\in\mathbb{Z}} K_n$ where K_n is compact.

Exercise. 1) Find a compact l-space X and $U \subseteq X$ such that U is not countable at ∞.

2) Every σ-compact, S_1 l-space X is homeomorphic to one of the following:

(a) Countable (or finite) discrete space.

(b) Cantor set.

(c) Cantor set minus a point.

d) Disjoint union of b) or c) with a).

Definition. $Refinement$ of a cover $\cup U_i = X$ is a cover $\{V_j\}$ such that for any j, we have that $V_j \subseteq U_i$ for some i.

Exercise. 1) Let $C \subseteq X$ be a compact subset of an l-space. Then any cover has an open compact disjoint refinement.

2) Let X be a countable at ∞ l-space, then any cover has an open compact disjoint refinement.

Distributions on l-spaces.

Definition. Let X be an l-space. A function f from X to the field will be called a *smooth function* if for every point $x \in X$ there is an open neighborhood U such that the restriction $f|_U$ is constant.

Proposition. *Let X be an l-space. Show that the smooth functions $C^\infty(X)$ separates the points in X. Assuming this exercise, the Stone-Weierstrass theorem implies that $C^\infty(X)$ is dense in $C(X)$.*

Proof. Let $x, y \in X$. As X is Hausdorff and having a basis of open compact, . there exists U_x and U_y compact and open. Set $f(U_x) = 0$ and $f(X/U_x) = 1$. Then $f(x) = 0$, and $f(y) = 1$. □

Definition. The functions with compact support, $C_c^\infty(X) \subset C^\infty(X)$, are called *Schwartz functions.* We denote them by $S(X)$. We also denote $Dist(X) = C_c^\infty(X)^* = S(X)^*$. We consider both spaces as vector spaces without topology.

Exercise. Let X be an l-space, show that $C_c^\infty(X)^*$ is a sheaf.

Remark. In $\mathbb{R}^n$, the Schwartz functions are the functions whose derivatives decrease faster than every polynomial, and $C_c^\infty(\mathbb{R}^n) \subset S(X) \subset C^\infty(\mathbb{R}^n)$. We will define them in the next lectures.

4.6. **Distributions supported on a subspace.** Recall that over $\mathbb{R}$, the description of distributions on a space X that are supported on Z is a little complicated (we did that using filtrations). Distributions on l-spaces behave much better.

Definition. Let X be an l-space, the *support of a distribution* $\xi \in S^*(X)$ is $Supp\xi$ = the smallest closed subset S such that $\xi|_{X \setminus S} = 0$.

Proposition. *Let $i : S^*(Z) \to S_Z^*(X)$ be the map induced by the restriction $Res : S(X) \longrightarrow S(Z)$. Then i is an inclusion.*

Proof. We prove it by showing the dual map $j : S(X) \to S(Z)$ is onto. Let $f \in S(Z)$. As f is locally constant and compactly supported, we may assume that Z is compact and has a covering by a finite number of open sets U_α (open in Z) with $f|_{U_\alpha} = c_\alpha$. Notice that each U_α, is of the form $U_\alpha = W_\alpha \cap Z$, where W_α is open in X . Therefore, $Z \subseteq \{W_\alpha\}$, and as Z is compact, we may refine $\{W_\alpha\}$ and get that $Z \subseteq \cup_i V_i$ when V_i open compact and $V_i \cap Z \subseteq W_\alpha \cap Z = U_\alpha$ for some α. Therefore we can extend f by defining $f(x) = c_\alpha$ if $x \in V_i \subseteq W_\alpha$ and zero otherwise. □

Proposition. *(Exact Sequence of an Open Subset). Let $U \subseteq X$ be open and set $Z = X \setminus U$. Then $0 \to S(U) \to S(X) \to S(Z) \to 0$ is exact.*

Proof. We showed that $S(X) \to S(Z)$ is onto, and it is clear that extension by zero $S(U) \to S(X)$ is injective. It is left to prove exactness in the middle. Let $f \in S(X)$ such that $f|_Z = 0$. As f is locally constant, there is an open set $V \supseteq Z$ such that $f|_V = 0$. This implies that f is supported on $Z^C = U$ and therefore $f|_U \in S(U)$. □

Corollary. *Let X be an l-space, and $Z \subset X$ a closed subspace. Then:*

1) The inclusion $i : S^*(Z) \to S^*_Z(X)$ is an isomorphism.

2) There is an exact sequence $0 \to S^*(Z) \to S^*(X) \to S^*(X \backslash Z) \to 0$.

Remark. Note that over $\mathbb{R}$, the map i is not onto . For example, for $Z := \{0\} \subset \mathbb{R}$, the derivatives $\delta_0^{(n)} \in S^*_Z(\mathbb{R}^n)$ but not in the image of i. Moreover, on $\mathbb{R}^n$we have an exact sequence:

$$0 \to S^*_Z(X) \to S^*(X) \to S^*(X \backslash Z).$$

Exercise. Let V be a vector space (maybe infinite-dimensional) over a field K, and $L \subset V$ a linear subspace. Show that $\forall f \in L^* \; \exists g \in V^* : g|_L \equiv f$. Use Zorn's lemma.

So far we showed two advantages of distributions on l-spaces over distributions on $\mathbb{R}^n$:

(1) Every distribution ξ supported on some $Z \subset X$ is also supported on a neighborhood of Z.
(2) The map $i : S^*(Z) \to S^*_Z(X)$ is onto.

Both these qualities can be achieved over $\mathbb{R}^n$ by switching from $C_c^\infty(\mathbb{R}^n)$ to real-valued Schwartz functions. A third advantage is:

Proposition. *Let X, Y be l-spaces. Given $f_1 \in S(X), f_2 \in S(Y)$, consider the bilinear map $\phi : S(X) \otimes S(Y) \to S(X \times Y)$ where $(\phi(f_1 \otimes f_2))(x, y) := f_1(x) \cdot f_2(y)$. Then ϕ is locally constant and an isomorphism of vector spaces.*

Proof. The locally constant property is easy to see by refinement of the open sets in X and Y . **Surjectivity:** let $f \in S(X \times Y)$. Then $f = \sum c_{U_i \times V_i}$ and by refining $\{U_i \times V_i\}$ we may assume that they are disjoint. Notice that each term $c_{U_i \times V_i} \in \phi(f_1, f_2)$ so we are done.

Injectivity: Assume that $\phi(\sum_i f_{1i} \otimes f_{2i}))(x;y) := \sum_i f_{1i}(x) \cdot f_{2i}(y) = 0$. We can assume that $\{f_{1i}\}$ are linearly independent and that $\{f_{2i}\}$ are non zero and that the sum is minimal with respect to those demands. If we take some y such that $f_{21}(y) \neq 0$ we get that for any $x \in X$, $\sum_i f_{1i}(x) \cdot f_{2i}(y) = 0$. This implies that $f_{1i}(x)$ are linearly dependent. Contradiction. Hence $f_{2i} \equiv 0$ and hence $f_{1i} \otimes f_{2i} \equiv 0$ and contradiction to the minimality of the summation. □

5. Lecture 5- Distribution with values on a vector space

Definition. Let F be a local field. V a vector space over F. We can define $C_c^\infty(X,V)$ to be the space of smooth functions with compact support from X to V, with the same convergence condition as in the usual $V = F$ case. Here the smoothness of a function is the usual coordinate-wise one.

Exercise. Prove that $C_c^\infty(X,V) \cong C_c^\infty(X) \otimes_F V$ as topological vector-spaces, the topology on $C_c^\infty(X) \otimes_F V$ is given by choosing a basis to identify V with F^n and then take the product topology on $C_c^\infty(X) \otimes_F F^n \cong_{can} (C_c^\infty(X))^n$. In particular, this topology is independent on a choice of a basis.

5.1. Smooth measures.

A measure has 2 equivalent definitions: A σ additive map from the σ-algebra of Borel subsets of X into $\mathbb{R}$. For us, the following definition is better:

Definition: Let X be a locally compact topological space. The space of *signed measures* on X is $C_c(X)^*$, i.e. a continuous functional on $C_c^\infty(X)$. A signed measure is a measure if it is non-negative on non-negative function.

As the space $C_c(X)$ is larger than $C_c^\infty(X)$, its dual is smaller. Specifically $C_c(X)^* \subseteq C_c^\infty(X)^*$, the inclusion is the dual of the obvious continuous map $C_c^\infty(X) \hookrightarrow C_c(X)$. Inside $C_c(X)^*$ there is a one dimensional space of Haar measures, which in this case is just the space of multiples of the Lebesgue measure.

Definition: Let V be a locally compact f.d vector space (If it is not finite dimensional then it can't be locally compact). The space of *Haar measures on* V, denoted $h_V \subseteq C_c(V)^*$, is the space of translation invariant measures.

The fact that this space is one dimensional is non-trivial, but the intuition is as follows: A Borel measure on X is determined by its value on cubes with sides parallel to the axes planes of rational side length, as they form basis of the topology. It is not hard to see that if the measure is translation invariant, the measures of these cubes are determined by the measure of the unit cube.

Exercise. Let $\xi \in C_c^\infty(X)^*$ which is translation invariant. Prove that ξ is a Haar measure. Note that $C_c^\infty(X)^* \supsetneq C_c(X)^*$so there might be other translation invariant functionals other then the Haar measure.

Definition. A measure μ on V is called a *smooth measure* if $\mu \in C^\infty(V)h_V$, i.e. $\mu = f(x)h$ where f is smooth and h is a Haar measure. We denote this space by $\mu_c^\infty(V)$. Note that by definition, $\mu_c^\infty(V) \simeq_{can} C_c^\infty(V, h_V) \simeq C_c^\infty(V) \otimes h_V$ canonically, and also $\mu_c^\infty(V) \simeq C_c^\infty(V)$ by choosing some Haar measure, but this isomorphism is not canonical.

5.2. **Generalized Functions Versus Distributions.** We are now in position to understand the difference between generalized functions and distributions.

A *distribution* on V is continuous functional on the space of smooth functions with compact support:

$$Dist(V) := C_c^\infty(V)^*$$

.A *generalized function* is a continuous functional on the space of smooth measures with compact support on V, i.e.

$$C^{-\infty}(V) := C_c^\infty(V, h_V)^*.$$

As functions can be integrated against smooth measures, thus we have a pairing $C_c^\infty(V, h_V) \times C_c^\infty(V) \overset{<,>}{\rightarrow} F$. Though we have the following picture:

$$\begin{array}{ccc} C_c^{-\infty}(V) & \overset{\simeq}{\Longleftrightarrow} & Dist(V) \\ j \uparrow & & i \uparrow \\ C_c^\infty(V) & \overset{\simeq}{\Longleftrightarrow} & \mu_c^\infty(V) \end{array}$$

And the diagonals are dual to each other. The inclusion $i : \mu_c^\infty(V) \hookrightarrow Dist(V)$ is via the pairing $C_c^\infty(V, h_V) \times C_c^\infty(V) \overset{<,>}{\rightarrow} F$, and the inclusion $j : C_c^\infty(V) \hookrightarrow C_c^{-\infty}(V)$ is defined by $f \longmapsto \varphi_f$ where $\varphi_f(\mu) = \int f d\mu$ for a smooth measure μ.

Exercise. $h_V \simeq Dist(V)^V$ or equivalently $Dist(V)^V$ is one dimensional, for any finite dimensional vector space V over a local field F.

Definition. We can also define generalized functions with value in a vector space, by either:

1) $C^{-\infty}(V, E) := C^{-\infty}(V) \otimes E$

2) $C^{-\infty}(V, E) := C_c^\infty(V, h_V \otimes E^*)^*$

and then $C^{-\infty}(V, h_V) := C^{-\infty}(V) \otimes h_V = C_c^\infty(V)^* = Dist(V)$.

Exercise. 1) Show that the two definition for $C^{-\infty}(V,E)$ are equivalent.

2) Define an embedding $C_c^{\infty}(V,E) \hookrightarrow C^{-\infty}(V,E)$.

5.3. **Some linear algebra.** Let V be an n dimensional vector space over a local field F. Let $\Omega^{top}(V)$ be the space of anti-symmetric n-forms on V . It is a one-dimensional space, and $\Omega^{top}(V) = \bigwedge^n(V^*)$.

Exercise. Let $\mathcal{B}$ be the space of bases of V . Show that $\Omega^{top}(V) = \{f : \mathcal{B} \to F : f(B_1) = det(M_{B_1}^{B_2})f(B_2)\}$ and also that $\Omega^{top}(V) = \{f : V^n \to F : f(Av_1, ..., Av_n) = det(A)f(v_1, ..., v_n)\}$.

Definition. If V is over $\mathbb{R}$, then we have two related spaces, the space of densities and the space of orientations:

$$Dens(V) = \{f : \mathcal{B} \longrightarrow \mathbb{R} : f(B_1) = \left|det(M_{B_1}^{B_2})\right| f(B_2)\}$$

And

$$Ori(V) = \{f : \mathcal{B} \longrightarrow \mathbb{R} : f(B_1) = sign(det(M_{B_1}^{B_2})) \cdot f(B_2)\}$$

Exercise. $\Omega^{top}(V) = Dens(V) \otimes Ori(V)$, via the tensor product of the natural maps $\Omega^{top}(V) \to Dens(V)$ and $\Omega^n(V) \longrightarrow Ori(V)$.

Note that this space of orientation is a linear space and not two points as one expect from orientation. On the other hand, we have two distinguished points in $Ori(V)$, the two functions with absolute value 1. These are the usual orientations we used to think of.

Exercise. $Dens(V) \simeq_{can} h_V$.

Definition. Let F be a local field with absolute value $|\,|$. We can define a functor $|\,|$on V from one dimensional vector spaces over F to one dimensional vector spaces over $\mathbb{R}$, by

$$|V| := \{f : V^* \longrightarrow \mathbb{R} | f(\alpha v) = |\alpha| f(v)\}$$

Exercise. 1) $|L \otimes M| = |L| \otimes |M|$.

2) $|\Omega^{top}(V)| \simeq Dens(V)$.

3) If $W \subseteq V$ then $h_W \otimes h_{V/W} \cong_{can} h_V$.

4) $W \subseteq V$, $\Omega^{top}(V) \simeq \Omega^{top}(W) \otimes \Omega^{top}(V/W)$

5) If $F = \mathbb{R}$, then $Ori(V) = Ori(W) \otimes Ori(V/W)$.

5.4. **Generalized Functions With Support on a Subspace.** Let $W \subseteq V$ be a linear spaces. We showed that over a non archimedean field F, $Dist_W(V) = Dist(W)$, and for $F = \mathbb{R}$ we have described the case of $V = \mathbb{R}^n$ and $W = \mathbb{R}^k$. The goal now is to describe the distributions on V supported on W for any $W \subseteq V$ linear spaces over $\mathbb{R}$. Recall that inside $Dist(V)$, there is a subspace $Dist_W(V)$ of distributions supported on W. We have defined a filtration $F^i_W(V)$ on $C^\infty_c(V)$ by

$$F^i_W(V) = \{f \in C^\infty_c(V) : Df|_W = 0, |D| \le i\}$$

and we have defined $F_{i,W}(V) \subseteq Dist_W(V)$ by

$$F_i(V)_W = \left(C^\infty_c(V)/F^i_W(V)\right)^* := \{\xi \in Dist(V) | \langle \xi, f\rangle = 0 \text{ for any } f \in F^i_W(V)\}$$

We denote $F_i(V)_W = F^i_W(V)^\perp$ where $Y^\perp := (X/Y)^*$. We want to describe $F_i(V)_W/F_{i-1}(V)_W$ in canonical terms, i.e. in a way invariant under diffeomorphisms preserving W.

Theorem. *We have a (V, W)-diffeomorphism preserving isormophism of vector spaces:*

$$F_i(V)_W/F_{i-1}(V)_W \cong_{can} C^\infty_c(W, Sym^i(W^\perp))^* \simeq Dist(W) \otimes Sym^i(V/W).$$

Observe that $Sym^i(W^\perp) = SymPoly(V/W, ..., V/W; \mathbb{R}) = \{f : V^i \longrightarrow \mathbb{R} | f|_{W\times V\times ...\times V} = 0\}$. The theorem is based on the following lemma:

Lemma. $F_i(V)_W/F_{i-1}(V)_W \cong (F^{i-1}_W(V)/F^i_W(V))^*$.

Proof. For $\phi \in F_i(V)_W$, $\phi|_{F^{i-1}_W(V)}$ vanish on $F^i_W(V)$, and we send it to the induced functional on $F^{i-1}_W(V)/F^i_W(V)$, denoted $\widetilde{\phi}$. This is an injective morphism, as if $\widetilde{\phi} = 0$ then $\phi|_{F^{i-1}_W(V)} = 0$ so $\phi \in F_{i-1}(V)_W$. surjectivity follows from Hahn-Banach theorem in the following way: any $\varphi \in (F^{i-1}_W(V)/F^i_W(V))^*$ can be extended to $\widetilde{\varphi} \in \left(C^\infty_c(V)/F^i_W(V)\right)^* = F_i(V)_W$. Therefore $[\widetilde{\varphi}] + F_{i-1}(V)_W \longmapsto \varphi$. □

Hence, in order to prove the theorem it will be sufficient to prove that $F^{i-1}_W(V)/F^i_W(V) \cong C^\infty_c(W, Sym^i(W^\perp))$. For this we will do the natural thing- attach to f its i-th derivatives. Explicitly, we define:

$$\Phi(f)(w)(v_1, ..., v_i) = \partial_{v_1}...\partial_{v_i} f(w).$$

It is well defined as f vanish identically on W, so this form kills all the tangential derivatives. It is one-to-one as if $\Phi(f) = 0$, then f vanish with all of its derivatives up to degree i so it is in $F^i_W(V)$.

Exercise. 1) prove that Φ is onto, hence an isomorphism.

2) Show that the isomorphism $F_i(V)_W/F_{i-1}(V)_W \cong_{can} C_c^\infty(W, Sym^i(W^\perp))^*$ is invariant with respect to diffeomorphism of (V, W).

3) Find $\xi \in Dist(V-W)$ s.t there is **no** $\eta \in Dist(V)$ such that $\eta|_{V-W} = \xi$. That is, the natural map $Dist(V) \longrightarrow Dist(V-W)$ is not onto.

For the generalized-functions case, we get the same result by twisting with Haar measures. Indeed,

$$F_i(V)_W/F_{i-1}(V)_W \cong C_c^\infty(W, Sym^i(W^\perp))^* = C^{-\infty}(W, Sym^i(W^\perp) \otimes h_W)$$

. Take $G_i(V)_W = G_i(V)_W \otimes h_V^* \subseteq C^{-\infty}(V)$. We get, by the compatibility of tensor and quotient,

$$G_i(V)_W/G_{i-1}(V)_W \cong C^{-\infty}(W, Sym^i(W^\perp) \otimes h_W) \otimes h_V^* \cong C^{-\infty}(W, Sym^i(W^\perp) \otimes h_W \otimes h_V^*)$$

But what is this (one dimensional) space $h_W \otimes h_V^*$?

Exercise. If $W \subseteq V$, then:

1) $h_W \otimes h_{V/W} \cong_{can} h_V$.

2) $h_V^* = h_{V^*}$.

From the exercise it follows that

$$h_W \otimes h_V^* = (h_W^* \otimes h_V)^* = (h_W^* \otimes h_W \otimes h_{V/W})^* = h_{V/W}^* = h_{(W^\perp)^*}^* = h_{W^\perp}.$$

Corollary. *By the above argument it follows that:*

$$G_i(V)_W/G_{i-1}(V)_W \cong C^{-\infty}(W, Sym^i(W^\perp) \otimes h_{W^\perp}).$$

6. Lecture 6- Manifolds

Definition. 1) Let X be a topological space. A cover $\{U_i\}$ is called $locally\, finite$, if for any $x \in X$ there is a neighborhood V such that V intersects only finite number of sets in the cover.

2) A topological space X is called *paracompact*, if any open cover has a refinement that is locally finite.

3) A $toplological\, manifold$ is a topological space X that is locally homeomorphic to $\mathbb{R}^n$, Hausdorff and paracompact.

Exercise. 1) Find a space X which is locally homeomorphic to $\mathbb{R}^n$ at every point and is paracompact but is not Hausdorff.

2) Find a space which is Hausdorff, locally isomorphic to $\mathbb{R}^n$ but is not paracompact.

We will now give a definition for (smooth) manifolds that is different then the usual definition in differential topology. We will use the following more general definition of sheaves of functions:

Definition. A *sheaf of (K-valued) functions* $\mathcal{F}$ on a topological space X is an assignment $U \longmapsto \mathcal{F}(U) \subseteq \{f : U \longrightarrow K | \, f \text{ is continuous}\}$ such that:

1) $\mathcal{F}(U)$ is an algebra with unity.

2) $Res^U_V \mathcal{F}(U) \subseteq \mathcal{F}(V)$ is the usual restriction $f \longmapsto f|_U$.

3) For every open cover $U = \bigcup_{i\in I} U_i$, if there exists a set of functions $\{f_i\} \in \mathcal{F}(U_i)$ s.t. : $f_i|_{(U_i \cap U_j)} \equiv f_j|_{(U_i \cap U_j)}$ for any $i, j \in I$ then there exists $f \in \mathcal{F}(U)$ s.t. $f|_{U_i} \equiv f_i$ for any $i \in I$.

A sheaf of functions on X will be denoted by a pair $(X, \mathcal{F})$

Remark. Note that the second demand implies the identity axiom.

Example. Function sheaves can be "continuous functions on X", "smooth functions on X".

Definition. 1) Let $(X, \mathcal{F}), (Y, \mathcal{G})$ be sheaves of functions. Then a morphism $\varphi : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ is a map $\varphi : X \longrightarrow Y$ such that $\forall g \in \mathcal{G}(U)$ we have that $g \circ \varphi|_{\varphi^{-1}(U)} \in \mathcal{F}(\varphi^{-1}(U))$. In other words, a map $\varphi^{\#} : \mathcal{G} \longrightarrow \varphi_* \mathcal{F}$.

2) A *smooth manifold* is a space with functions $(X, C^\infty(X))$, where X is a topological manifold and for every point $x \in X$ there is a open neighborhood U such that $(U, C^\infty(X)|_U) \simeq (\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ as sheaves of functions, that is maps $\varphi : U \longrightarrow \mathbb{R}^n$, $\varphi^{\#} : (\mathbb{R}^n, C^\infty(\mathbb{R}^n)) \longrightarrow \varphi_* C^\infty(X)|_U$ and $\psi : \mathbb{R}^n \longrightarrow U$, $\psi^{\#} : C^\infty(X)|_U \longrightarrow \psi_* C^\infty(\mathbb{R}^n))$ such that $\varphi = \psi^{-1}$ and $(\psi \circ \varphi)^{\#} = Id_{C^\infty(X)|_U}$, $(\varphi \circ \psi)^{\#} = Id_{C^\infty(\mathbb{R}^n)}$.

Remark. The usual definition of manifolds adds an "atlas" to the structure of X: an open cover $X = \bigcup_{i\in I} U_i$ with diffeomorphism $\phi_i : U_i \to \mathbb{R}^n$. But we also demand that$\phi_i \circ \phi_j^{-1}$ is differentiable, so it looks like an "extra" demand with respect to the definition above. If we look closely, we see that a pair of isomorphisms

$\varphi_i : (U_i, C^\infty(U_i)) \longrightarrow (\mathbb{R}^n, C^\infty(\mathbb{R}^n))$, and $\varphi_j : (U_j, C^\infty(U_j)) \longrightarrow (\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ implies that

$$\left(\varphi_i \circ \varphi_j^{-1}|_{U_i \cap U_j}\right)^{\#} : (\mathbb{R}^n, C^\infty(\mathbb{R}^n))|_{\varphi_j(U_i \cap U_j)} \longrightarrow \left(\varphi_i \circ \varphi_j^{-1}\right)_* (\mathbb{R}^n, C^\infty(\mathbb{R}^n))|_{\varphi_i(U_i \cap U_j)}$$

is an isomorphism. In particular, by the following exercise, we can deduce that $\varphi_i \circ \varphi_j^{-1}|_{U_i \cap U_j}$ is smooth and actually a diffeomorphism. Therefore the 2 above definitions for smooth manifolds are equivalent.

Exercise. 1) Show that $C^\infty(\mathbb{R}^n; \mathbb{R}^k) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R}^k : f^*(\mu) \in C^\infty(\mathbb{R}^n)\, \forall \mu \in C^\infty(\mathbb{R}^k)\}$.

2) A map $f : M \longrightarrow N$ is a smooth map of manifolds iff it is a morphism of ringed spaces (sheaves of smooth functions).

A theorem by Whitney shows that every n-dimensional manifold can be embedded in $\mathbb{R}^{2n+1}$.

6.1. **Tangent space of a manifold.** There are several "equivalent definition" for a tangent space of a smooth manifold M at a point $x \in M$. We will give a categorical definition and then we will give several proofs of existence that they will all be equivalent.

Definition. A tangent space of a smooth manifold M at a point $x \in M$ is a functor $Tan : (M, x) \longmapsto T_x M$ from pointed smooth manifolds to vector spaces satisfy the following conditions:

1) $(V, 0) \longmapsto V$.

2) If $f_1, f_2 : (M, x) \longrightarrow \mathbb{R}$ satisfy that $(f_1 - f_2)(y) = o(\|x - y\|)$ for any norm $\|\ \|$ on a manifold, then $Tan(f_1) = Tan(f_2)$.

3) If $U \hookrightarrow M$ is an open embedding, then $Tan((U, x) \hookrightarrow (M, x))$ is an isomorphism.

There are several structures that satisfy the above conditions:

(1) $T_x(M) := \{\gamma : ((-1, 1), 0) \to (M, x)\}$ modulo the relation $\gamma_1 \sim \gamma_2$ iff exists a neighborhood U of x and a isomorphism $\phi : U \to \mathbb{R}^n$ s.t. $(\phi \circ \gamma_1)'(x) - (\phi \circ \gamma_2)'(x)$. It is easy to check that this definition doesn't depend on the choice of (ϕ, U)
(2) $T_x(M) = \{d : C^\infty(M) \to \mathbb{R} |\ d \textit{ is linear}, d(f \cdot g) = df \cdot g(x) + f(x) \cdot dg\}$. This is the space of derivations.
(3) Define $m_x := \{f \in C^\infty(M) \,|\, f(x) = 0\}$, and take $T_x(M) := (m_x/m_x^2)^*$.

Exercise. Show the definitions are equivalent.

Definition. Now let $\phi : M \to N$ be smooth. The *differential of* ϕ *in* $x \in M$ is a linear map $d_x\phi : T_x(M) \to T_{\phi(x)}(N)$ that $d_x(\phi)(\gamma) := \phi \circ \gamma$.

Exercise. Show that given manifolds M, N, K and maps $\phi : M \to N, \psi : N \to K, \nu = \psi \circ \phi : M \to K$, the differentials satisfy $d_x(\nu) \equiv d_{\phi(x)}(\psi) \circ d_x(\phi)$.

6.2. **Type of maps of smooth manifolds.**

Definition. Let $\phi : M \to N$ be a smooth map between manifolds.

*ϕ is an *immersion* if $d_x\phi$ is one-to-one.

* ϕ is a *submersion* if $d_x\phi$ is onto.

* ϕ is a *local isomorphism* or *étale* if $d_x\phi$ is one-to-one and onto.

* ϕ is an *embedding* if it's an immersion and there is a homeomorphism $M \cong \phi(M)$.

* ϕ is a *proper map* if for every compact K, the preimage $\phi^{-1}(K)$ is compact. In particular, fibers are compact in M.

* ϕ is a *cover map* if for $x \in N$ there exists a neighborhood $U \subseteq M$, such that $\phi|_{\phi^{-1}(U)} : \phi^{-1}(U) \to U$ is a diffeomorphism, and is a composition $\phi^{-1}(U) \to U \times D \to U$ for a discrete set D.

Example. 1) Let $\phi : [-1, 1] \to \mathbb{R}^2$ be a smooth path that slows to a stop in $\phi(0) = (0, 0)$, but spends no time in $(0, 0)$. That is, all the derivatives are zeroed $\phi^{(n)}(0) = 0$, but $\phi(x) \neq 0$ for all x in some neighborhood $[-\epsilon, \epsilon]$. Such a ϕ is one-to-one around 0, but is not an immersion at 0.

2) An immersion isn't necessarily one-to-one. An example is a self-intersecting path $\phi : \mathbb{R} \to \mathbb{R}^2$ with constant speed.

3) Let L, D be finite dimensional linear spaces. The differential of a map $\phi \in Hom(L, V)$ is ϕ itself. Thus, a one-to-one ϕ will be an immersion, an onto ϕ will be a submersion, and an isomorphism of linear space will be an étale.

Exercise. 1) Find a $\phi : M \to N$ which is a one-to-one immersion, but isn't an embedding.

2) Show that every proper map which a one-to-one immersion is a closed embedding.

3) Show that a proper map which is an étale is a cover map, and that a cover map with finite fibers is proper and étale.

Definition. A *fibration* is a map $X \xrightarrow{p} Y$, where locally $p^{-1}(U) \simeq U \times Z$ for $U \subseteq Y$ and some topological vector space Z.

Exercise. A proper submersion is a fibration.

Definition. Given a submanifold $X \subseteq M$, and an embedding $i : X \to M$, we define the *normal bundle at a point* $x \in M$ to be $N_x(M) := i^*(TM)/TX$. Similarly, the *conormal bundle* is $CN_x(M) := (N_x(M))^*$.

Example. For $M = S^2$ the normal bundle at a point will give the normal vector to it. It will be isomorphic to the trivial bundle on M.

6.3. **Analytic manifolds and Vector bundles.** We would like to introduce to more important structures: an analytic F- manifold (for any local field F) and a real vector bundle.

Definition. 1) An analytic F-manifold is a space M which is locally isomorphic to F^n together with a sheaf of functions

$$An(U) = \{f : U \to F : \forall x \in U, \exists r > 0 \text{ s.t. } f_{|B_r(x)}(y) = \sum_{\vec{k}\in\mathbb{N}^n} a_{\vec{k}}(x-y)^{\vec{k}}\},$$

where $B_r(x)$ is the ball of radius r around x, and $\vec{k}$ is a multi-index, thus $(x-y)^{\vec{k}} = \prod_{i=0}^{n}(x_i - y_i)^{k_i}$.

2) A $smooth\,analytic\,manifold$ is a ringed space (of functions) $(X, \mathcal{F})$ locally isomorphic to (F^n, An).

Remark. We don't have partition of unity in analytic manifolds. If an analytic function zeroes in some neighborhood, it must be the zero function.

Example. There exists singular analytic manifolds, and in particular any singular affine algebraic variety.

Definition. Let M be a smooth manifold or a p-adic analytic manifold. A real vector bundle over M is a tuple (E, p) where E is a topological space and $p : E \to M$ is a continuous surjection such that:

1) For every $x \in M$ we have a structure of a finite dimensional real vector space on $p^{-1}(x) = V_x$.

2) For every $x \in M$ there exists an open $x \in U$ and a local trivialization $\varphi_U : V_x \times U \to p^{-1}(U)$ where φ_U is a homeomorphism (or diffeomorphism if M is a real smooth manifold) and $p \circ \varphi_U(v, x) = x$ for all $v \in V_x$.

3) The maps $v \mapsto \varphi_U(v,x)$ are linear isomorphisms.

If $E \simeq V \times M$ we say (E,p) is a *trivial bundle* over M.

Example. 1) (exercise) The ***Mobius strip*** is homeomorphic to $I \times S^1$. By extending each segment I to $\mathbb{R}$, we can define a bundle over the manifold S^1. This way, the points in E are pairs (θ, x), where x runs over the points of the line of angle $0.5 \cdot \theta$. Define the vector bundle above rigorously and show it is not diffeomorphic to the bundle $S^1 \times \mathbb{R}$. You can assume the Mobius strip isn't diffeomorphic to the S^1.

2) The tangent bundle of $M = S^1$ is $TS^1 \simeq S^1 \times \mathbb{R}$. The tangent space at any point is one-dimensional, and changes smoothly as we "walk" on the circle. However, on $M = S^2$ the tangent bundle will not be isomorphic to $S^2 \times \mathbb{R}^2$. This holds since every vector field on S^2 vanishes ("you can't comb a hedgehog").

Definition. Let (M,E) be a vector bundle. Given neighborhoods U, V , consider $\varphi_V^{-1} \circ \varphi_U : (U \cap V) \times \mathbb{R}^k \longrightarrow (U \cap V) \times \mathbb{R}^k$. We can write $\varphi_V^{-1} \circ \varphi_U(x,v) = (x, g_{UV}(v))$ where $g_{UV} \in GL(\mathbb{R}^k)$. The maps g_{UV} are called *transition functions.*

Notice that the set of transition functions g_{UV}, satisfy the cocycle conditions $g_{UU}(x) = Id$, $g_{UV}(x) g_{VW}(x) = g_{UW}(x)$. Conversely, given a fiber bundle $(E, X, \pi, \mathbb{R}^k)$ with a $GL(\mathbb{R}^k)$ cocycle acting in the standard way on the fiber $\mathbb{R}^k$, there is associated a vector bundle. This is sometimes taken as the definition of a vector bundle.

Definition. Let E_1, E_2 be two vector bundles over M. The direct sum $E_1 \oplus E_2$ is defined as follows:

Given a bundle $\pi_1 : E_1 \longrightarrow M$ and $\pi_2 : E_1 \longrightarrow M$, and a collection of trivializations $\phi_i^1 : \pi_1^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^k$, $\phi_i^2 : \pi_2^{-1}(V_i) \longrightarrow V_i \times \mathbb{R}^k$, by refining the covers we may assume that $V_i = U_i$. Now define $E_1 \oplus E_2 := \bigsqcup_{m \in M} E_{1,m} \oplus E_{2,m}$ as a set where $E_{i,m} = \pi_i^{-1}(m)$. The map $\pi : E_1 \oplus E_2 \longrightarrow M$ is defined by the canonical projection. We define the topology on $E_1 \oplus E_2$ by the trivializations $\psi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^{2k}$ by $\psi_i(m,(v,w)) = (m, \phi_i^1(m,v), \phi_i^2(m,u))$. It is easily seen that the transition functions $\psi_j \circ \psi_i^{-1} : U_i \cap U_j \times \mathbb{R}^{2k} \longrightarrow U_i \cap U_j \times \mathbb{R}^{2k}$ satisfy

$$\psi_j \circ \psi_i^{-1}(m,(v,u)) = (m, \phi_j^1 \left(\phi_i^1\right)^{-1}(m,v), \phi_j^2 \left(\phi_i^2\right)^{-1}(m,u))$$

and as ϕ^1 and ϕ^2 satisfy the cocycle conditions, so does ψ. This defines a structure of a vector bundle.

Exercise. Find non-isomorphic bundles E, E', such that $E \oplus F \cong E' \oplus F$ for some bundle F (**Hint**: use vector bundles over S^2).

Similarly, we can define tensor products of bundles, multilinear n-forms, and their absolute value and sign.

Exercise. 1) Let $\Phi : Vect^n \longrightarrow Vect^m$ be a "smooth" functor, i.e, $\Phi(T)$,$T \in Hom(V,W) \simeq \mathbb{R}^{n^2}$ is a smooth map from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{m^2}$. Define functor $\widetilde{\Phi} : VecBun(M)^n \longrightarrow VecBun(M)^m$.

2) For vector bundles E_1, E_2, define the following notions: (you can, and advised to, use part 1)

(a) E_1^*.

(b) $E_1 \oplus E_2$.

(c) $E_1 \otimes E_2$.

(d) For an embedding $\varphi : E_1 \hookrightarrow E_2$, define E_2/E_1.

(e)$\bigwedge^k(E_1)$, $Sym^k(E_1)$.

(f) In the real/complex case, define $Dens(E_1)$.

Definition. 1) Let M be a smooth manifold, we can define its *density bundle* by $D_M = |\Omega^{top}(TM)|$, that is the density bundle of the tangent bundle.

2) Let X be an F analytic manifold, we define its density bundle by $D_X = |\Omega^{top}(TX)|$.

6.4. **sections of a bundle.** A set theoretic *section* of a function $f : X \to Y$ is a function $g : Y \to X$ s.t. $g \circ f \equiv id$. For example, for $f : \mathbb{R}^2 \to \mathbb{R}$ which is the projection $f(x,y) := x$ a section can be $g(x) := (x, sinx)$. This a just be a (continuous) choice of representatives of fibers.

In our case, sections of bundles can help us define many basic concepts. For example:

- A section of the tangent bundle is a *vector field*.
- A section of kth exterior power of the cotangent bundle is a *differential form* of degree k.
- A section of the density bundle is called a *density*.
- A section of the orientations bundle is an *orientation* on a manifold.

Exercise. 1) Show the every manifold has a Riemannian metric, i.e , an inner product on tangent spaces

$$<,>_p\colon T_pM \times T_pM \to \mathbb{R}$$

which varies smoothly.

2) Let M be a smooth n-dimensional Riemannian manifold, that is a smooth real manifold with a Riemannian metric. Construct explicitly a density over M, that is a smooth section of the density bundle over M. The density should respect coordinate changes, and be the standard density when M is a linear space with the standard inner product.

Remark. We don't always have top differential forms on a manifold, and the Mobius strip is a counter-example. However, we can always define densities.

Since a density over a space gives us a measure on it, we can thus define integrals over manifolds.

6.5. **Equivalent description of vector bundles.**

Definition. Let V be a finite dimensional vector space and X a topological space. We define the *constant sheaf* $\underline{V}_X$ to be the sheafification of the constant presheaf, which assigns to every open set in X the vector space V. We say that a sheaf $\mathcal{F}$ over X is *locally constant* if for every $x \in X$ there exists an open $x \in U_x$ and a finite dimensional vector space V_x such that $\mathcal{F}_{|U_x} \simeq \underline{V_x}_{U_x}$.

Exercise. 1) Let V be a finite dimensional vector space and X a topological space. Show that $\underline{V}_X(U)$ consists of the locally constant functions from U to V.

2) Show that if X is a σ-compact ℓ-space then every locally constant sheaf $\mathcal{F}$ where $\mathcal{F}_x \simeq \mathcal{F}_y$ for all $x, y \in X$ is isomorphic to the constant sheaf.

Definition. *Leray* sheaf on X is a local homeomorphism $p : E \longrightarrow X$.

Theorem. *The definition of a Leray sheaf is equivalent to the Grothendieck definition of a sheaf.*

Proof. Given a Leray sheaf p we define a Grothendieck sheaf $\mathcal{F}(U) := \{\text{continuous sections } U \longrightarrow p^{-1}(U)\}$. For the other direction, given a Grothendieck sheaf $\mathcal{F}$, we define $E = \bigsqcup_{x \in X} \mathcal{F}_x$ with the natural projection map $p : E \longrightarrow X$. We define a basis for the topology of E by $U_{s,V} = \{(x, (s)_x) : x \in V\}$ where $V \subseteq X$ is open and $s \in \mathcal{F}(V)$. $\square$

Exercise. 1) Complete the proof by showing that those functors induce an equivalence of categories.

2) Show that covering spaces correspond to locally constant sheaves, and that a covering space is trivial exactly when it corresponds to a constant sheaf.

3) Give an example for a locally constant sheaf arising from a covering space which is not constant.

7. Distribution on analytic/smooth manifolds

Definition. Let E be an F-analytic one dimensional vector bundle over an F-analytic manifold X. Define a real vector bundle $|E|$ as follows. As a set define $|E| := \{(x,v)|x \in X, v \in |E_x|\}$ and define a topology by giving $\mathbb{C}$ the discrete topology, so locally $E|_U \simeq U \times F$ and $|E|\,|_U \simeq U \times |F| \simeq U \times \mathbb{C}$. Hence, a base for the topology is $V_{i,U,\alpha} = \varphi_i(U \times \{\alpha\})$ where $\varphi_i : U \times \mathbb{C} \longrightarrow |E|\,|_U$ and $\alpha \in \mathbb{C}$.

Remark. Note that $\widetilde{p} : |E| \longrightarrow X$ is a local homeomorphism as $V_{i,U,\alpha} \simeq U$ as a topological space. Hence $\widetilde{p}$ is a Leray sheaf. Its corresponding Grothendieck sheaf is $\mathcal{F}(U) := \{\text{continuous sections } U \longrightarrow \widetilde{p}^{-1}(U)\}$. This is a locally constant sheaf $\underline{\mathbb{C}}_X$over X.

Definition. We can now define density bundle over an F-analytic manifold X in two ways:

Def 1 (Leray): $D_X := |\Omega^{top}(X)|$.

Def 2 (Grothendieck):

$D_X(U) := \{\mu \in Mesures(U) | \forall \varphi \in \mathcal{O}_F^n \longrightarrow U, \text{there exists } f \in C^\infty(\mathcal{O}_F^n) \text{ such that } \mu = \varphi_*(f \cdot Haar)\}$

Lemma. *Let* $\varphi : F^n \longrightarrow F^n$ *be analytic diffeomorphism and* $f \in C_c(F^n)$. *Then* $h_{F^n}(f) =: \int f dx = \int (f \circ \varphi) dx \cdot |det(D_x \varphi)|$.

Exercise. Show that the above definitions are equivalent.

7.1. Smooth sections of a vector bundle.

In this subsection we assume that $F = \mathbb{R}$ and we are dealing with smooth manifolds.

Definition. We define

$C_c^\infty(M,E) := \{f : M \longrightarrow E \text{ such that } \pi \circ f = Id_M \text{ and } \exists K \text{ compact such that } f|_{K^C}(m) = (m,0)\}$

Recall that $C_c^\infty(\mathbb{R}^n,\mathbb{R}^k)=\underrightarrow{lim}C_{K_m}^\infty(\mathbb{R}^n,\mathbb{R}^k)$ where K_m is an increasing sequence of compact sets. We will now define a topology on $C_c^\infty(M,E)$ using the topology on $C_c^\infty(\mathbb{R}^n,\mathbb{R}^k)$:

Case 1- The trivial case: $M\simeq\mathbb{R}^n$ and $E\simeq\mathbb{R}^n\times\mathbb{R}^k\longrightarrow\mathbb{R}^n$. Note that continuous sections from $\mathbb{R}^n$ to $\mathbb{R}^n\times\mathbb{R}^k$ just means a function in $C_c^\infty(\mathbb{R}^n,\mathbb{R}^k)$. Hence we give $C_c^\infty(M,E)$ the topology of $C_c^\infty(\mathbb{R}^n,\mathbb{R}^k)$.

Exercise. Show that the above definition is well defined, i.e, doesn't depend on the isomorphism $M\simeq\mathbb{R}^n$ and $E\simeq\mathbb{R}^n\times\mathbb{R}^k\longrightarrow\mathbb{R}^n$. In other words, show that:

(a) Given a diffeomorphism $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ then it induces a homeomorphism $\varphi^*:C_c^\infty(\mathbb{R}^n;\mathbb{R}^n\times\mathbb{R}^k)\longrightarrow C_c^\infty(\mathbb{R}^n;\mathbb{R}^n\times\mathbb{R}^k)$.

(b) Given a smooth map $\psi\in C^\infty(\mathbb{R}^n;GL_k(\mathbb{R}))$ we have that $\psi_*:C_c^\infty(\mathbb{R}^n;\mathbb{R}^n\times\mathbb{R}^k)\longrightarrow C_c^\infty(\mathbb{R}^n;\mathbb{R}^n\times\mathbb{R}^k)$ is a homeomorphism.

Case 2- General case: We can choose small enough $\{U_i\}$ such that $M=\cup U_i$ where $\varphi_i:U_i\xrightarrow{\simeq}\mathbb{R}^n$ and also $\psi_i:E|_{U_i}\xrightarrow{\simeq}\mathbb{R}^n\times\mathbb{R}^k$ (an isomorphism of vector bundles). We have a surjective map

$$\varphi:\bigoplus_{i\in I}C_c^\infty(U_i,E|_{U_i})\twoheadrightarrow C_c^\infty(M,E)$$

by summation (surjectivity follows from partition of unity). Hence we can define a quotient topology according to the map φ, that is, we define a set $U\subseteq C_c^\infty(M,E)$ to be open if $\varphi^{-1}(U)$ is open in $\bigoplus_{i\in I}C_c^\infty(U_i,E|_{U_i})$ (with the direct sum topology).

Exercise. Prove that this "definition" is well defined, i.e, show that that the definition doesn't depend on the cover U_i. Reduce to showing that $\bigoplus_{i\in I}C_c^\infty(U_i,\mathbb{R}^k)\longrightarrow C_c^\infty(\mathbb{R}^n,\mathbb{R}^k)$ is open. The full proof for this exercise is at "Tirgul 6".

Now we will give a different description for the topology of $C_c^\infty(\mathbb{R}^n)$. At first observe that $f\in C_c^\infty(\mathbb{R}^n)$ iff for any $g\in C^\infty(\mathbb{R}^n)$, gf is bounded. Now let $D\in Diff(\mathbb{R}^n)$- differential operators on $C_c^\infty(\mathbb{R}^n)$. Define a seminorm $\|f\|_D$ by $sup|D(f)|$. We get that $C_c^\infty(\mathbb{R}^n)$ is an inverse limit of Banach spaces B_D where each B_D is the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to $\|\ \|_D$.**(Verify with Rami).**

Definition. $Diff(M)$ is an operator on $C^\infty(M)\longrightarrow C^\infty(M)$ such that for any (or some) cover $\cup U_i=M$ such that $\varphi_i:U_i\longrightarrow\mathbb{R}^n$ we have that $\varphi_i^{-1}\circ D\circ\varphi_i\in Diff(\mathbb{R}^n)$.

Now we would like to define $Diff(C^\infty(M,E),C^\infty(M,E'))$. Again we divide it into cases:

Case 1- E,E' **is trivial:** $E\simeq M\times\mathbb{R}^k$ and $E'\simeq M\times\mathbb{R}^{k'}$. Then $Diff(C^\infty(M,E),C^\infty(M,E'))\simeq Diff(C^\infty(M)^k,C^\infty(M)^{k'})$ and this is isomorphic (as vector spaces) to $k\times k'$ matrices with values in $Diff(C^\infty(M))$.

Exercise. Show that the definition of differential operator $D\in Diff(C^\infty(M,E),C^\infty(M,E'))$ for $E\simeq M\times\mathbb{R}^k$ and $E'\simeq M\times\mathbb{R}^{k'}$ doesn't depend on the isomorphisms.

Case 2- **the general case:** Let $A\in Hom(C^\infty(M,E),C^\infty(M,E'))$. Then we say that $A\in Diff(C^\infty(M,E),C^\infty(M,E'))$ iff:

* For any $f_{1,2}\in C^\infty(M,E)$ such that $f_1|_U=f_2|_U$, then $Af_1|_U=Af_2|_U$.

* If $E'|_U$ is trivializable then $A|_U\in Diff(U,E|_U,E'|_U)$.

Definition. Second definition for the topology on $C_c^\infty(M,E)$: For any $D\in Diff(C^\infty(M,E),C^\infty(M,E))$ define $\|f\|_D=sup\,|D(f)|$ (choose some norm on E). Define the topology on $C_c^\infty(M,E)$ as

$$C_c^\infty(M,E)=\varinjlim_D(C_c^\infty(M,E),\|f\|_D).$$

Exercise. Given a manifold M and a vector bundle E over it show that the two definitions of the topology on $C_c^\infty(M;E)$ are equivalent (one defined via taking a cover of M and trivialization of E and the other through differential operators).

8. Distributions over geometric objects

Definition. 1) We define distributions on smooth sections by $Dist(M,E):=C_c^\infty(M,E)^*$.

2) We define generalized sections on a smooth vector bundle by $C^{-\infty}(M,E)=Dist(M,E^*\otimes D_M)$, where D_M is the density bundle.

We don't have a natural injection from $C_c^\infty(M,E)$ to $C_c^\infty(M,E)^*$ but we do have a natural injection $i:C_c^\infty(M,E)\hookrightarrow C^{-\infty}(M,E)$ as follows: Let $\mu\in C_c^\infty(M,E^*\otimes D_M)$ and $f\in C_c^\infty(M,E)$. Note that $f\otimes\mu\in C_c^\infty(M,E^*\otimes E\otimes D_M)$ (that is, $f\otimes\mu(m)=f(m)\otimes\mu(m)$). Note that we have a natural map $q:C_c^\infty(M,E^*\otimes E\otimes D_M)\longrightarrow C_c^\infty(M,D_M)$ and a natural map $\int:C_c^\infty(M,D_M)\longrightarrow\mathbb{C}$ by integrating on M according to the measure defined by the section of the density bundle. Hence $\langle i(f),\mu\rangle=\int_M q(f\otimes\mu)$. Therefore, the definition of generalized sections indeed generalizes smooth sections.

Exercise. Let M, N be either smooth or an F-analytic manifolds and let X, Y be l-spaces. Show that:

1) $\overline{C_c^\infty(M)}^w = C^{-\infty}(M)$.

2) $C_c^\infty(M) \otimes C_c^\infty(N) = C_c^\infty(M) \times C_c^\infty(N)$ and $C_c^\infty(X) \otimes C_c^\infty(Y) = C_c^\infty(X) \times C_c^\infty(Y)$

3) Find an example such that $C_c^\infty(X)^* \otimes C_c^\infty(Y)^* \ncong C_c^\infty(X \times Y)^*$.The same for (smooth/analytic) manifolds (Hint: consider $X = Y = \mathbb{Z}$.)

4) Let $E_{1,2}$ be complex vector bundles over $M_{1,2}$, then $C_c^\infty(M_1 \times M_2, E_1 \boxtimes E_2) = C_c^\infty(M_1, E_1) \otimes_{\mathbb{C}} C_c^\infty(M_2, E_2)$.

Definition. Let X be an l-space and $\mathcal{F}$ a sheaf over X. Define $\mathcal{F}_c(X)$ to be the space of *compactly supported global sections* of $\mathcal{F}$, that is, $s \in \mathcal{F}(X)$ such that $s|_{K^C} = 0$ outside some compact K. Define $C_c^\infty(X, \mathcal{F}) := \mathcal{F}_c(X)$.

Theorem. *Let $i : Z \hookrightarrow X$ be l-spaces. Then:*

1) $Dist(X, \mathcal{F})|_Z \simeq Dist(Z, \mathcal{F}|_Z) = i^*(\mathcal{F})$.

2) We have:

$$0 \longrightarrow Dist(Z, \mathcal{F}|_Z) \longrightarrow Dist(X, \mathcal{F}) \longrightarrow Dist(U, \mathcal{F}|_U) \longrightarrow 0.$$

We now want to prove the following important theorem:

Theorem. *Let $N \subseteq M$ a closed (real) submanifold and E a bundle over M. Then there is a canonical filtration $F_i \subseteq Dist_N(M, E)$ (supported on N) such that:*

i) F_i is locally exhaustive, i.e, $\bigcup F_i$ ***is locally*** *$Dist_N(M, E)$.*

ii) $F_i / F_{i-1} \simeq Dist(N, E|_N \otimes Sym^i(CN_N^M))$.

In order to prove the theorem, we would like to define the notion of "vanishing of kth derivative of a smooth section $f \in C_c^\infty(M, E)$". The only problem is that the notion of kth derivative depend on the chart defined on M so it is not well defined. Fortunately, the notion of "vanishing kth derivative" is well defined as the following exercise shows:

Exercise. Let $f \in C^\infty(\mathbb{R}^n, \mathbb{C})$ with $f^{(i)}(0) = 0$ for every $|i| < k$, and $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a diffeomorphism such that $\varphi(0) = 0$. Furthermore let $g \in C^\infty(\mathbb{R}^n, \mathbb{C}^\times)$ be a smooth function, and set $\widetilde{f}(x) = f \circ \varphi^{-1}(x) g(x)$.

(a) Show that:

$$\left(\frac{\partial^k}{\partial v_1 ... \partial v_k}\widetilde{f}\right)(0) = \left(\frac{\partial^k}{\partial\left((D\varphi)v_1\right)...\partial\left((D\varphi)v_k\right)}f\right)(0)g(0)$$

(b) Part (a) might not be true if $f^{(i)}(0) \neq 0$ for some $|i| < k$.

As a consequence of this exercise, given any $f \in C_c^\infty(M,E)$ such that f vanishes with $k-1$ derivatives, we can define $D_x^k f : T_xM \times ...T_xM \longrightarrow E_x$ by

$$D_x^k f(\xi_{1,i},...,\xi_{k,i}) = \left(\frac{\partial^k}{\partial\xi_{1,i}...\partial\xi_{k,i}}(f\circ\varphi_i^{-1})\right)(0)$$

where φ_i is a local chart and$\xi_{1,i} = (\varphi_i \circ \gamma_1)'(0)$ is some tangent vector. If we choose a different chart φ_j we get that

$$D_x^k f(\xi_{1,j},...,\xi_{k,j}) = \left(\frac{\partial^k}{\partial\xi_{1,j}...\partial\xi_{k,j}}(f\circ\varphi_j^{-1})\right)(0) = \left(\frac{\partial^k}{\partial\xi_{1,j}...\partial\xi_{k,j}}(f\circ\varphi_i^{-1}\circ\varphi)\right)(0)$$

where $\varphi := \varphi_i \circ \varphi_j^{-1}$. By the discussion above, we get that

$$\left(\frac{\partial^k}{\partial\xi_{1,j}...\partial\xi_{k,j}}(f\circ\varphi_j^{-1})\right)(0) = \left(\frac{\partial^k}{\partial(D\varphi)\xi_{1,j}...\partial(D\varphi)\xi_{k,j}}(f\circ\varphi_i^{-1})\right)(0)$$

But

$$D_x\varphi(\xi_{1,j}) = D_x\varphi\cdot(\varphi_i\circ\gamma_1)'(0) = (\varphi\circ\varphi_i\circ\gamma_1)'(0) = \xi_{1,i}$$

So $D_x^k f(\xi_{1,j},...,\xi_{k,j}) = D_x^k f(\xi_{1,i},...,\xi_{k,i})$ and the definition is well defined. We can now proof the theorem:

Proof. (of Theorem) Note that we can identify $D_x^k f \in Sym^k(T_x^*M)\otimes E_x$. Let $N \subseteq M$ be a submanifold. Define:

$$F_N^i(C_c^\infty(M,E)) = \{f \in C_c^\infty(M,E) | \forall x \in N,\ f \text{ vanishes with } i-1 \text{ derivatives}\}$$

Recall that for locally, $M|_U \simeq \mathbb{R}^n$ and $N|_{U\cap N} \simeq \mathbb{R}^k$, and we showed that $F_W^{i-1}(V)/F_W^i(V) \cong C_c^\infty(W, Sym^i(W^\perp))$ using the map $f \longmapsto D_x^k f$. Hence by applying a small generalization we get:

$$F_N^i/F_N^{i-1} \simeq C_c^\infty(N, E|_N \otimes_{\mathbb{C}} Sym^i(CN_N^M)).$$

This gives a canonical filtration $F_i \subseteq Dist_N(M,E)$ such that

$$F_i/F_{i-1} \simeq \left(F_N^i/F_N^{i-1}\right)^* \simeq C_c^\infty(N, E|_N \otimes_{\mathbb{C}} Sym^i(CN_N^M))^* = Dist(N, E|_N \otimes_{\mathbb{C}} Sym^i(CN_N^M)).$$

∎

Corollary. *We have that* $Gr_i(C^{-\infty}(M,E)_N) = C^{-\infty}(N, E|_N \otimes D_M^*|_N \otimes Sym^i(N_N^M)) \otimes D_L)$.

Proof. We have that:

$$Gr_i(C^{-\infty}(M,E)_N) = Gr_i(Dist_N(M, E^* \otimes D_M))$$

$$\simeq Dist(N, E^*|_N \otimes D_M|_N \otimes Sym^i(CN_N^M)) = C^{-\infty}(N, E|_N \otimes D_M^*|_N \otimes Sym^i(N_N^M)) \otimes D_L).$$

□

8.1. **Operations on generalized functions.** At this subsection we assume X, Y are either l-spaces, analytic F-manifolds (with or without complex bundles over them), or smooth manifolds.

Definition. Let $\varphi : X \longrightarrow Y$ be a map. We can define *Pullback* of the space of functions by $\varphi^* : C^\infty(Y) \longrightarrow C^\infty(X)$ by $\varphi^*(f) = f \circ \varphi$. It is easy to see that if φ is proper then $\varphi^* : C_c^\infty(Y) \longrightarrow C_c^\infty(X)$. This give rise to a definition of *Pushforward* of distributions $\varphi_* : Dist(X) \longrightarrow Dist(Y)$ by $\varphi_*(\xi)(f) := \xi(\varphi^*(f)) = \xi(f \circ \varphi)$.

Note that if φ is not proper then we have $\varphi_* : Dist(X)_{prop} \longrightarrow Dist(Y)$ where $Dist(X)_{prop} := \{\xi \in Dist(X) |\, \varphi|_{supp(\xi)}$ is proper$\}$. We would like to define it by $\varphi_*\xi(f) = \xi(f \circ \varphi)$. But $f \circ \varphi$ is not compactly supported. Therefore we choose a cutoff function ρ such that $\rho|_{supp(\xi)} = 1$ and $\rho|_{U^C} = 0$ where U is a small neighborhood of $supp(\xi)$ such that $\varphi|_{\overline{U}}$ is proper (This is a hard task to find such a function). Hence we can define $\varphi_*\xi(f) := \xi(\rho \cdot (f \circ \varphi))$. Note that $supp(\rho \cdot (f \circ \varphi)) \subseteq supp(\rho) \cap \varphi^{-1}(supp(f)) \subseteq \varphi|_{supp(\rho)}^{-1}(supp f)$. Since $\varphi|_{supp \rho}$ is proper, and f is compactly supported, this is well defined. The definition clearly doesn't depend on the choice of ρ.

Recall that for vector spaces we had $Dens(V) \simeq haar_V$. Hence we identify/define the space of smooth measures $\mu_c^\infty(X)$ as a the space of smooth sections of the density bundle $C_c^\infty(X, D_X)$. Note that we can define $\varphi_* : C_c^\infty(X, D_X) \longrightarrow Dist(X)$ by $\varphi_*(\mu)(f) = \int_X f d\mu$.

Exercise. $\varphi_*(Dist_{comp}(X)) \subseteq Dist_{comp}(Y)$.

Proposition. *If* $\varphi : X \longrightarrow Y$ *is a submersion, then:*

1) $\varphi_*(\mu_c^\infty(X)) \subseteq \mu_c^\infty(Y)$.

2) In addition, if $\phi = f \cdot |\omega_X|$ *and* $\varphi_*(f \cdot |\omega_X|) = g \cdot |\omega_Y|$*, where* $|\omega_X|, |\omega_Y|$ *are non-vanishing densities on* X, Y *then* $g(y) = \int_{\varphi^{-1}(y)} f \frac{|\omega_X|}{|\varphi^* \omega_Y|}$ *where* $\frac{|\omega_X|}{|\varphi^* \omega_Y|} \in D_{\varphi^{-1}(y)}$

satisfy that $|\omega_X| \simeq \frac{|\omega_X|}{|\varphi^*\omega_Y|} \otimes |\omega_Y|$ *according to the natural isomorphism* $(D_X)_x \simeq (D_{\varphi^{-1}(y)})_x \otimes (D_Y)_{\varphi(x)}$.

Proof. 1) Case 1: $X = F^n, Y = F^m$ and $\varphi : F^n \longrightarrow F^m$ is the natural projection $\varphi(x_1, ..., x_n) = x_1, ..., x_m$. Recall that $haar_X \simeq haar_Y \otimes haar_{X/Y}$ or equivalently $D_X|_{(x_1,...,x_n)} \simeq D_Y|_{(x_1,...,x_m)} \otimes D_{X/Y}|_{(\overline{x_1},...,\overline{x_n})}$, where $(\overline{x_1}, ..., \overline{x_n}) = x \in X/Y$. Let $\phi \in C_c^\infty(X, D_X)$ and note that $\phi = f \cdot d\mu_X$ where $f \in C_c^\infty(X)$ and μ_X is the canonical Haar measure (taking the value 1 on the unit ball), so we can write $\mu_X = \mu_Y \otimes \mu_{X/Y}$. By definition, for any $g \in C_c^\infty(Y)$ we have:

$$\langle \varphi_*(\phi), g\rangle = \langle \phi, g \circ \varphi\rangle = \int_X f \cdot (g \circ \varphi) d\mu_X = \int_Y \int_{X/Y} f \cdot (g \circ \varphi) d\mu_Y \otimes \mu_{X/Y}.$$

Since $g \circ \varphi(x_1, ..., x_n) = g(x_1, ..., x_m)$ depends only on Y so we have

$$\langle \varphi_*(\phi), g\rangle = \int_Y \left(\int_{X/Y} f \cdot d\mu_{X/Y} \right) \cdot g d\mu_Y = \int_Y \widetilde{f} \cdot g d\mu_Y$$

where $\widetilde{f} \in C_c^\infty(Y)$. Hence $\varphi_*(\phi)$ is a smooth function.

General case: $\varphi : X \longrightarrow Y$ is a submersion. Write $Y = \cup V_j$ and then $X = \cup U_i$ such that $\varphi(U_{i_j}) \subseteq V_j$. For any i, j such that $\varphi(U_i) \subseteq V_j$ we can choose isomorphisms $\tau_i : U_i \simeq F^n$ and $\psi_j : V_j \simeq F^m$ such that $\psi_j \circ \varphi \circ \tau_i^{-1}$ is the natural projection $F^n \longrightarrow F^m$. Hence $\left(\psi_j \circ \varphi \circ \tau_i^{-1}\right)_* (\mu_c^\infty(F^n) \subseteq (\mu_c^\infty(F^m)$ and $\varphi_*(C_c^\infty(U_i, D_{U_i}) \subseteq C_c^\infty(V_j, D_{V_j})$. By partition of unity, we can write $\phi = \sum f_i d\mu_i$ where $f_i d\mu_i \in C_c^\infty(U_i, D_{U_i})$ (this is a finite sum as ϕ is compactly supported). Finally, observe that:

$$\varphi_*(\phi) = \varphi_*(\sum f_i d\mu_i) = \sum \varphi_*(f_i d\mu_i) = \sum g_i d\mu_i.$$

Each $g_i d\mu_i$ is a smooth distribution, so also the sum $\sum g_i d\mu_i$.

2) Since $\varphi$is a submersion then for any $\varphi(x) = y \in Y$ $\varphi^{-1}(y)$ is a submanifold of X and we have the following exact sequence:

$$0 \longrightarrow T_x\varphi^{-1}(y) \longrightarrow T_x(X) \longrightarrow T_{\varphi(x)}(Y) \longrightarrow 0.$$

Therefore $T_x(X) = T_x\varphi^{-1}(y) \oplus T_{\varphi(x)}(Y)$ and hence also

$$0 \longrightarrow T^*_{\varphi(x)}(Y) \longrightarrow T^*_x(X) \longrightarrow T^*_x\varphi^{-1}(y) \longrightarrow 0.$$

This give rise to the following equality: $(D_X)_x = (D_{\varphi^{-1}(y)})_x \otimes (D_Y)_{\varphi(x)}$ by $(\psi \otimes \tau) \longmapsto \phi$. Precisely, we choose basis $v_1, ..., v_m$ of $T_x\varphi^{-1}(y)$ and a complement basis $v_{m+1}, ..., v_n$ such that $d\varphi(v_{m+1}, ..., v_n)$ is a basis of $T_{\varphi(x)}(Y)$. We now define

$\phi(v_1 \wedge .... \wedge v_n) := \psi((v_1 \wedge .... \wedge v_m) \cdot \tau(d\varphi(v_{m+1} \wedge .... \wedge v_n)$. It can be checked that the isomorphism doesn't depend on the choice of the basis.

We first reduce the problem to a small neighborhood, write $Y = \cup V_j$ and then $X = \cup U_i$ such that $\varphi(U_{i_j}) \subseteq V_j$. For any i, j such that $\varphi(U_i) \subseteq V_j$ we can choose isomorphisms $\tau_i : U_i \simeq F^n$ and $\psi_j : V_j \simeq F^m$ such that $\psi_j \circ \varphi \circ \tau_i^{-1}$ is the natural projection.

We need to prove that $\varphi_*(f\,|\omega_X|)(h) = f\,|\omega_X|\,(h \circ \varphi) = g \cdot |\omega_Y|\,(h)$ where g as in the above formula. Construct partition of unity $f = \sum f_i$. Then it is enough to prove the claim for $f_i\,|\omega_X|$ as then:

$$\varphi_*(f\,|\omega_X|)(h) = \varphi_*(\sum f_i\,|\omega_X|)(h) = \sum \int_Y g_i h\,|\omega_Y|$$

where $g_i(y) = \int_{\varphi^{-1}(y)\cap U_i} f_i \cdot \eta = \int_{\varphi^{-1}(y)} f_i \cdot \eta$. As $g = \sum g_i$ we have that $g(y) = \int_{\varphi^{-1}(y)} f \cdot \eta$ as required.

The case of projection $\varphi : F^n \longrightarrow F^m$ was solved at a). Hence it is enough to reduce to this case. Using the fact that for diffeomorphism, pushforward is inverse to pullback, we get:

$$\psi_j \circ \varphi_*(f_i\,|\omega_X|) = \psi_j \circ \varphi \circ \tau_{i*}^{-1}((\tau_i^{-1})^*\,(f_i\,|\omega_X|)) = \psi_j \circ \varphi \circ \tau_{i*}^{-1}(f_i \circ \tau_i^{-1} \cdot \left|(\tau_i^{-1})^* \omega_X\right|) = \widetilde{g_i}\,\left|(\psi_j^{-1})^* \omega_Y\right|$$

where $\widetilde{g_i}(x) = \int_{\tau_i \circ \varphi^{-1} \circ \psi_j^{-1}(x)} f_i \circ \tau_i^{-1} \left|\frac{(\tau_i^{-1})^* \omega_X}{(\varphi \circ \tau_i^{-1})^* \omega_Y}\right|$. Denote $g_i := \widetilde{g_i} \circ \psi_j$. So $\widetilde{g_i} = g_i \circ \psi_j^{-1}$. Hence $\varphi_*(f_i\,|\omega_X|) = g_i\,|\omega_Y|$. Also

$$g_i(y) = \widetilde{g_i}(\psi_j(y)) = \int_{\tau_i^{-1} \circ \varphi^{-1}(y)} f_i \circ \tau_i^{-1} \left|\frac{(\tau_i^{-1})^* \omega_X}{(\varphi \circ \tau_i^{-1})^* \omega_Y}\right| = \int_{\varphi^{-1}(y)} f_i \left|\frac{\omega_X}{\varphi^* \omega_Y}\right|.$$

□

Definition. By the proposition, the map $\varphi_* : C_c^\infty(X, D_X) \longrightarrow C_c^\infty(Y, D_Y)$ gives a pullback $\varphi^* : C^{-\infty}(Y) \longrightarrow C^{-\infty}(X)$.

Exercise. Let $\varphi : X \longrightarrow Y$ be a submersion. We defined a pullback $\varphi^* : C^\infty(Y) \longrightarrow C^\infty(X)$ both by $\varphi^*(f) = f \circ \varphi$ and by first defining $\varphi^* : C^{-\infty}(Y) \longrightarrow C^{-\infty}(X)$ via the definition for compactly supported smooth measures, and then by restricting to $C^\infty(Y)$. Show that the two definitions coincide.

We can also generalize the push and pull of functions and distribution to bundles:

Definition. 1) Let $\varphi : X \longrightarrow Y$ and $\pi : E \longrightarrow Y$ a bundle. Define $\varphi^*(E) := \{(x, e) \in X \times E | \varphi(x) = \pi(e)\}$ as a bundle over X with the natural projection to X.

2) We can now define pullback of sections $\varphi^* : C^\infty(Y, E) \longrightarrow C^\infty(X, \varphi^*(E))$ and pushforward of distributions $\varphi_* : Dist(X, \varphi^*(E))_{prop} \longrightarrow Dist(Y, E)$.

3) Let $\varphi : X \longrightarrow Y$. Define $\varphi^!(E) := \varphi^*(E) \otimes \varphi^*(D_Y^*) \otimes D_X$.

Proposition. *Let $\varphi : X \longrightarrow Y$ be a submersion. Then $\varphi_*\left(C_c^\infty(X, \varphi^*(E) \otimes D_X)\right) \subseteq C^\infty(Y, E \otimes D_Y)$. In particular, This implies that $\varphi_*\left(C_c^\infty(X, \varphi^!(E))\right) \subseteq C^\infty(Y, E)$.*

Proof. As in the proof of the last proposition, we may reduce to the case where $\varphi : X \longrightarrow Y$ is the natural projection, $X = F^n$, $Y = F^m$, $E \simeq F^m \times F^k$ is trivial and as a consequence $\varphi^*(E) = F^n \times F^k$. We may do the reduction since the notion of "smoothness" of a distribution is local. Let $\phi = f d\mu \in C_c^\infty(X, \varphi^*(E) \otimes D_X)$. Then we have for any $g \in C^\infty(Y, E)$,

$$\langle \varphi_*(\phi), g\rangle = \langle \phi, g \circ \varphi\rangle = \int_X f \cdot (g \circ \varphi) d\mu_X = \int_Y \left(\int_{X/Y} f \cdot d\mu_{X/Y} \right) \cdot g d\mu_Y = \int_Y \widetilde{f} \cdot g d\mu_Y = \langle \widetilde{f} d\mu_Y, g\rangle$$

so $\varphi_*(\phi)$ is smooth. □

9. Fourier transform

Definition. Let G be a locally compact Hausdorff abelian group. Define its Pontryagin dual by,

$$G^\vee = \{\chi : G \to \mathrm{U}_1(\mathbb{C}) = S^1 \subseteq \mathbb{C} | \, \chi(g_1 g_2) = \chi(g_1)\chi(g_2), \ \chi \text{ is cts}\}$$

The topology on $G^\vee$ is the compact open topology, i.e. a sub-basis for the topology is comprised of sets $M(K, V) = \{\chi \in G^\vee : \chi(K) \subseteq V\}$ where $K \subseteq G$ is compact and $V \subseteq S^1$ is open.

Theorem. *1) Let G be a locally compact, Hausdorff abelian group, then $G^\vee$ is a locally compact Hausdorff abelian group.*

2) Let G be a locally compact, Hausdorff abelian group. Show that if G is compact then $G^\vee$ is discrete, and that if G is discrete then $G^\vee$ is compact.

Proof. At the tutorial session. □

Theorem. *For a locally compact abelian group G, we have that that the natural map $\varphi : G \longrightarrow G^{\vee\vee}$ defined by $g \longmapsto \varphi_g$, where $\varphi_g(\chi) = \chi(g)$, is an isomorphism $G^{\vee\vee} \simeq G$.*

Proposition. *Let G be a locally compact, Hausdorff abelian group, and $H \leq G$ a closed subgroup. Then:*

1) Pontryagin duality is a contravariant endofunctor in the category of locally compact abelian groups.

2) Show that $H^\vee \simeq G^\vee/H^\perp$ where $H^\perp = \{\chi \in G^\vee : \chi(h) = 1\ \forall h \in H\}$, and that if H and G are vector spaces then this is a homeomorphism.

Proof. At the tutorial session. □

Example. 1) For any finite abelian group G we have that $G \simeq \hat{G}$.

2) The dual of $\mathrm{U}_1(\mathbb{C}) = S^1$ is $\mathbb{Z}$.

3) For $G = \hat{\mathbb{R}} \simeq \mathbb{R}$.

Exercise. Let V be a topological vector space over a local field F. Then $V^* \otimes_F F^\vee \simeq V^\vee$.

Definition. Let G be a locally compact Hausdorff abelian group. The map $\mathcal{F} : \mu_c(G) \longrightarrow C(G^\vee)$ defined by $\mathcal{F}(\mu)(\chi) = \int \chi d\mu$ is called $Fourier\, transform$.

Exercise. 1) $\mathcal{F}(\mu)$ is continuous.

2) Let G be a locally compact abelian group. Show that for $\eta \in \mu_c^\infty(G)$ and $g \in G$:

(a) $\mathcal{F}(sh_g(\eta))(\chi) = \chi(g)\mathcal{F}(\eta)(\chi)$ for all $\chi \in G^\vee$.

(b) $\mathcal{F}(\chi\eta) = sh_{\chi^{-1}}(\mathcal{F}(\eta))$ for all $\chi \in G^\vee$.

Definition. Let X_1, X_2 be locally compact T.V.S and let $\mu_1 \in \mu_c^\infty(X_1)$,$\mu_2 \in \mu_c^\infty(X_2)$. We can define the tensor product of such measures $\mu_1 \boxtimes \mu_2 \in \mu_c^\infty(X_1 \times X_2)$.In addition, If $X_1 = X_2 = G$, then we can also define convolution of measures by $\mu_1 * \mu_2 := m_*(\mu_1 \boxtimes \mu_2)$ where $m : G \times G \longrightarrow G$.

Fact. $\mathcal{F}(\alpha * \beta) = \mathcal{F}(\alpha) \cdot \mathcal{F}(\beta)$.

Definition. Let V be a f.d vector space over a local field F. Define $Schwartz\, functions$ on V by:

1)$\mathcal{S}(V) = C_c^\infty(V)$, i.e. locally constant functions on V, in case F is non-archimedean.

2) $\mathcal{S}(V) = \{f \in C^\infty(V) | \forall i \in \mathbb{N}^n, p \in F[V],\, sup\left|\partial^i f \cdot p(x)\right| < \infty\}$, if F is archimedean. In other words it is the space of rapidly decreasing smooth functions on V.

Exercise. 1) Let $\mathcal{F}$ be a local field, show that $\mathcal{F}(\mathcal{S}(V; Haar(V)) \subseteq \mathcal{S}(V^\vee)$. In particular, for the subspace $\mu_c^\infty(V) \subset \mathcal{S}(V; Haar(V)$ we have that $\mathcal{F}(\mu_c^\infty(V)) \subset S(V^\vee)$.

2) Give $\mu_c^\infty(V)$ the subspace topology of $\mathcal{S}(V; Haar(V)$ and $\mathcal{S}(V^\vee)$ the natural Fréchet topology. Show that $\mathcal{F} : \mu_c^\infty(V) \longrightarrow \mathcal{S}(V^\vee)$ is continuous with respect to these topologies.

3) Show that $\mu_c^\infty(V)$ is dense in $\mathcal{S}(V; Haar(V)$ and that $\mathcal{S}(V; Haar(V)$ is complete and deduce that $\mathcal{F} : \mathcal{S}(V; Haar(V)) \longrightarrow \mathcal{S}(V^\vee)$ is continuous.

Definition. 1) Let $S(V)$ be the space of Schwartz functions on V. Then $\xi \in S^*(V)$ is called the*tempered distribution* and $\xi \in \mathcal{G}(V) := S^*(V, Haar(V))$ is called a *tempered generalized function.*

2) Finally, we now can define the Fourier transform on tempered distributions via duality:

$$\widetilde{\mathcal{F}} : S^*(V^\vee) \to \mathcal{G}(V) := S^*(V, Haar(V)).$$

Choosing $V := V^\vee$ we get $\widetilde{\mathcal{F}} : S^*(V) \to \mathcal{G}(V^\vee)$.

Theorem. *The definition of Fourier transform of distributions is consistent with the definition for functions. In other words* $\widetilde{\mathcal{F}}|_{\mathcal{S}(V;Haar(V)} = \mathcal{F}$.

Proof. Let $f(x) \cdot dx \in \mathcal{S}(V; Haar(V))$ and $g(\chi) \cdot d\chi \in \mathcal{S}(V^\vee; Haar(V^\vee)$. Then by definition,

$$\langle \widetilde{\mathcal{F}}(f(x) \cdot dx), g(\chi) \cdot d\chi \rangle := \langle f(x) \cdot dx, \mathcal{F}(g(\chi) d\chi) \rangle = \int_V f(x) \mathcal{F}(g(\chi) \cdot d\chi)(x) dx$$

where $\mathcal{F}(g(\chi) \cdot d\chi)(x) := \int_{\hat{V}} \chi(x) g(\chi) d\chi$. Therefore we have:

$$\int_V f(x) \mathcal{F}(g(\chi) \cdot d\chi)(x) dx = \int_V f(x) \int_{\hat{V}} \chi(x) g(\chi) d\chi dx = \int_{\hat{V}} \left(\int_V \chi(x) f(x) dx \right) g(\chi) d\chi$$

$$= \int_{\hat{V}} \left(\mathcal{F}(f)(\chi) \right) g(\chi) d\chi = \langle \mathcal{F}(f(x) \cdot dx), g(\chi) \cdot d\chi \rangle.$$

□

In the following argument we would like to present the Fourier transform as a unitary operator. For this we will first need to define a pairing between $Haar(V)$ and $Haar(V^\vee)$. Given $\alpha \in Haar(V)$ and $\beta \in Haar(V^\vee)$ we can define such a paring as follows. We choose $f \in C_c^\infty(V^\vee)$ such that $f(0) = 1$ and then define $\langle \alpha, \beta \rangle := \langle \mathcal{F}(\alpha), f \cdot \beta \rangle$.

Exercise. 1) This definition is well defined. That is, given some other $g \in C_c^\infty(V^\vee)$ such that $g(0) = 1$, show that $\langle \mathcal{F}(\alpha), (f - g) \cdot \beta \rangle = 0$.

2) Show that $h_{V^\vee} \simeq_{can} h_V^*$.

Definition. We can now define a map $\mathcal{F}_n : S^*(V, h_V^{\otimes n}) \longrightarrow S^*(V^\vee, h_{V^\vee}^{\otimes(1-n)})$ by using the pairing $h_{V^\vee} \simeq_{can} h_V^*$and identifying $S^*(V, h_V^{\otimes n})$, $S^*(V^\vee, h_{V^\vee}^{\otimes(1-n)})$ with $S^*(V) \otimes h_V^{\otimes -n}$ and $S^*(V^\vee, h_{V^\vee}) \otimes (h_{V^\vee})^{\otimes n}$ respectively. The identification between $S^*(V, h_V^{\otimes n})$ and $S^*(V) \otimes h_{V^\vee}^{\otimes n}$ is as follows. Given $\xi \otimes \beta \in S^*(V) \otimes h_{V^\vee}^{\otimes n}$ and $f \cdot \alpha \in$

$S(V, h_V^{\otimes n})$ we set $\xi \otimes \beta \longmapsto \xi\beta$ where $\xi\beta(f\alpha) = \xi(f\langle\beta, \alpha\rangle)$. Under this notation, we have that $\mathcal{F}_0 : S^*(V) \longrightarrow S^*(V^\vee, h_{V^\vee})$ is the Fourier transform.

Proposition. *We have that* $\mathcal{F}_1 \circ \mathcal{F}_0 = flip$ *where* $flip(\xi)(f(x)\mu) = \xi(f(-x)\mu)$.

Proof. Note that $span\{\delta_x\}$ is a dense subspace of $S^*(V)$ in the weak topology. Hence it is enough to show that $\mathcal{F}_1 \circ \mathcal{F}_0(\delta_a) = \delta_{-a}$. Note that $\mathcal{F}_0(\delta_0)(f\beta) := \langle\delta_0, \mathcal{F}_0(f\beta)\rangle = \int_{V^\vee} f d\beta$. Hence $\mathcal{F}_0(\delta_0) = 1$.

Note that $\mathcal{F}_1 : S^*(V^\vee, h_{V^\vee}) \longrightarrow S^*(V)$ is defined by identifying : $S^*(V^\vee, h_{V^\vee})$ with $S^*(V^\vee) \otimes h_V$. Under this identification, $1 := (1 \cdot \mu_1) \otimes \mu_2$where $1 \cdot \mu_1 \in S^*(V^\vee)$, $\mu_2 \in h_V$ and $\langle\mu_1, \mu_2\rangle = 1$. Now given $f \in S(V)$, $\mathcal{F}_1((1 \cdot \mu_1) \otimes \mu_2)(f) = \widetilde{\mathcal{F}}(1 \cdot \mu_1)(f\mu_2) = f(0)$ so $\mathcal{F}_1 \circ \mathcal{F}_0(\delta_0) = \delta_0$.

Notice that $\mathcal{F}_0(Sh_a(\delta_0)) = \chi(a)$ (as a function of χ) and $\mathcal{F}_1(\chi(a))$. By continuity of $\mathcal{F}_0$ and $\mathcal{F}_1$ this implies that $\mathcal{F}_1 \circ \mathcal{F}_0 = flip$.(Need to finish) □

Definition. Let V be a vector space over F and let $\chi : F^\times \longrightarrow K^\times$ be a group homomorphism. We can define

$$\chi(V) := \{\varphi : V^* \longrightarrow K^\times | \varphi(\alpha f) = \chi(\alpha)\varphi(f)\}.$$

Example. If $\chi = Square : F^\times \longrightarrow F^\times$ by $\chi(a) = a^2$, then $Square(V) := \{\varphi : V^* \longrightarrow K | \varphi(\alpha f) = \alpha^2\varphi(f)\}$. Note that $Square(V) \simeq_{can} V \otimes V$ if V is one dimensional by $v \otimes w \longmapsto \varphi_v \cdot \varphi_w$. Note that given $\psi \in V^*$ we have $\varphi_v \cdot \varphi_w(\psi) = \psi(v) \cdot \psi(w)$ and $\varphi_v \cdot \varphi_w(a\psi) = a\psi(v) \cdot a\psi(w) = a^2\varphi_v \cdot \varphi_w(\psi)$.

Definition. Let V be a one dimensional vector space over $\mathbb{R}$.

1) A *positive structure* on V is a non trivial subset $P \subseteq V$ such that $\mathbb{R}_{\geq 0} \cdot P = P$.

2) If V has a positive structure, we can define

$$V^\alpha := |V|^\alpha = \{\varphi : V^* \longrightarrow \mathbb{R}^\times | \varphi(\beta f) = |\beta|^\alpha \cdot \varphi(f)\}.$$

Exercise. 1) Let $V/\mathbb{R}$ be a 1-dimensional vector space with a positive structure. Show that:

(a) $V \simeq_{can} |V|$.

(b) $V^{\alpha+\beta} \simeq_{can} V^\alpha \otimes V^\beta$ where $\alpha, \beta \in \mathbb{Q}^\times$.

2) Deduce that $h_V^\alpha \otimes h_V^\beta$.

Definition. We now can finally define

$$\mathcal{F}_\alpha : S^*(V, h_V) \longrightarrow S^*(V^\vee, h_{V^\vee}^{1-\alpha})$$

for $\alpha \in \mathbb{Q}$. In particular, choosing $\alpha = 1/2$ we have:

$$\mathcal{F}_{1/2} : S^*(V, h_V^{1/2}) \longrightarrow S^*(V^\vee, h_{V^\vee}^{1/2})$$